TWO-PHASE FLOW BALANCE EQUATIONS

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1. Introduction

This course is a first introduction to two-phase flow modelling. Its prerequisites are a good knowledge of tensorial calculus and a basic knowledge of the distribution theory, with of course a good knowledge of classical fluid mechanics. The aim of this course is to present in a detailed manner the balance equations governing two-phase flows, which are useful to the numerical prediction of such flows. Such a course cannot be exhaustive due to the richness of the subject, but we have tried to give to the students the main tools which are necessary to master before entering in the numerical tools devoted to two-phase flow studies. These tools can be classified in two main categories. The first one which has emerged in the past decades is based on the RANS approach (RANS means Reynolds Averaged Navier-Stokes). This category is particularly useful for engineering applications where only the large scales (or tendencies) of the flow fields are desired. The second category is grouped into the acronym DNS (for Direct Numerical Simulation). It is completely different from the first since all the flow details (in space as well as in time) are solved numerically. The first category of tools is based on averaged balance equations, in opposition to the second category which is based on local instantaneous (unaveraged) ones. The major difficulty of the RANS approach is the closure problem posed by the averaged equations. During the averaging process, a lot of information has been lost (since the small flow details are not accessible in the solution). However, the mean (average) effect of the lost details on the averaged quantities is not negligible at all. As a consequence, the physicist must provide closure laws for a certain number of unknown terms appearing in the averaged equations. Therefore, the main difficulty of the RANS approach is essentially of mathematical and physical nature. The difficulties of the DNS approach are completely different. Here, the closure issue is inexistent since the local instantaneous balance equations are solved directly, without any kind of averaging. The difficulty appears essentially in the large amount of flow details which are necessary to calculate. Very powerful computers are necessary, with large quantities of memory. Even with such powerful computers, the flows which are amenable to simulate are quite restricted in terms of their Reynolds number, number of mobile interfaces (like bubbles or droplets interfaces) and so on… In addition, special tracking algorithms are often necessary to simulate accurately these mobile interfaces. So we can say that the major difficulties encountered in the DNS approach are essentially numerical and of data processing nature.

The plan of this course is the following one. The local instantaneous balance equations are presented in section 2. These equations are useful in a two-fold manner. First, they are the basic equations solved by the DNS tools. Second, they also form the theoretical basis to develop the averaged equations which are solved by the RANS tools. The local instantaneous (unaveraged) equations are sometimes called the microscopic equations (or equations valid at the microscopic level of description) and the averaged equations are sometimes called the macroscopic ones. The averaged balance equations are derived in the sections 3 and 4. Section 3 presents the so-called *two-fluid* model. The two-fluid model constitutes a very general mathematical frame where the equations are written for the two phases in a symmetrical way, without assuming any particular interfacial configuration (or flow regime).

Section 4 is devoted to *dispersed* two-phase flows, which groups bubbly, droplet and particulate flows. One of the two phases is assumed to be dispersed in a large number of inclusions (or particles or globules) into the other phase which is called the continuous (or carrier) phase. Due to the obvious dissymmetry of such kinds of flows, the averaged equations can be written is a dissymmetrical manner, reflecting the flow dissymmetry. The dispersed phase is described in a manner analogous to the molecules in the context of the kinetic theory of gases. Several advantages can be gained by using this kind of description: the equations for the dispersed phase are easier to derive and to interpret since they resemble to the equations governing a single particle. The price to pay is a lost of generality and the difficulty to make the connection with the other (continuous) phase which is always treated in the context of the two-fluid model. Due to the fact that the equations for the dispersed phase are treated in a different manner than the equations for the continuous phase, this approach is called *hybrid*. The section 5 is devoted to the simplified study of various special cases. At this time, only the case of the terminal velocity of a rising bubble in a liquid has been presented. The last section 6 gives a comparison of the equations used in the NEPTUNE_CFD code to the exact equations derived in the previous sections. This gives an example of numerical application to the reader in the context of the RANS approach. The major assumptions made by the NEPTUNE-CFD team are listed and the simplified equations are presented.

2. Local instantaneous balance equations

2.1. Topological equations

Let:

$$F(\underline{x},t)=0 \tag{2.1}$$

be the geometrical equation defining the different interfaces in the flow. Let F be positive in phase 1 and F be negative in phase 2. The Phase Indicator Function (PIF) is a binary function which can be defined as:

$$\chi_1(\underline{x},t)=1-\chi_2(\underline{x},t)=Y(F(\underline{x},t)) \tag{2.2}$$

where Y is the Heaviside distribution. The unit vector normal to the interface and directed outward from phase k (k = 1,2) can be defined classically as (Aris, 1962):

$$\underline{n}_2=-\underline{n}_1=\nabla F/|\nabla F| \tag{2.3}$$

Let $\underline{w}$ be the velocity field associated to the interfacial surface. As F is identically zero for all points located on the interface, its convective time derivative at the velocity $\underline{w}$ is nil:

$$\frac{\partial F}{\partial t}+\underline{w}.\nabla F=0 \tag{2.4}$$

From (2.3)-(2.4), one can deduce the normal displacement speed of the interface (Delhaye, 1981):

$$\underline{w}.\underline{n}_2 = -\underline{w}.\underline{n}_1 = -\frac{\partial F / \partial t}{|\nabla F|} \tag{2.5}$$

One important remark is that two different velocity fields $\underline{w}$ differing only through their tangential component $\underline{w}_t = \underline{w} - (\underline{w}.\underline{n})\underline{n}$ give rise to the same interface motion according to Eq. (2.4). Therefore, the normal velocity component is the only one to be related unambiguously to the surface motion. From their definitions (2.2), one can deduce the following expressions for their spatial and time derivatives:

$$\begin{aligned} &\nabla\chi_1 = -\nabla\chi_2 = \delta(F)\nabla F \\ &\frac{\partial \chi_1}{\partial t} = -\frac{\partial \chi_2}{\partial t} = \delta(F)\frac{\partial F}{\partial t} \end{aligned} \tag{2.6}$$

where δ is the Dirac distribution, which is the derivative of the Heaviside distribution Y. From (2.4) and (2.6), one can deduce the following topological equation:

$$\frac{\partial \chi_k}{\partial t} + \underline{w}.\nabla\chi_k = 0 \quad k = 1,2 \tag{2.7}$$

From relations (2.3) and $(2.6)_1$, on can also deduce:

$$-\underline{n}_k.\nabla\chi_k = \delta(F)|\nabla F| \equiv \delta_I \quad \Leftrightarrow \quad \nabla\chi_k = -\underline{n}_k\delta_I \tag{2.8}$$

where δ_I is a Dirac distribution having the different interfaces as a support. It is called a *local instantaneous interfacial area concentration* by Kataoka (1986).

Let us finish this section by remarking some interesting properties of the PIF. As they are binary functions, they verify that:

$$\begin{aligned} &{\chi_K}^n = \chi_K \\ &\chi_1\chi_2 = 0 \end{aligned} \tag{2.9}$$

As it will be seen later, the averaged fractions of presence of the two phases are defined as the averages of the corresponding PIF ($\alpha_K = <\chi_K>$ whatever the kind of averaging operator denoted by < >). However, the two averaged fractions of presence α_K are *not* binary functions; hence do *not* verify relations like (2.9). This is an important difference between local-instantaneous and averaged quantities and has some consequences on the corresponding balance equations. The advantage of the relations (2.9) verified by the PIF is that we can write, for any two quantities A_k and B_k characterizing phase k:

$$(\chi_K A_K)(\chi_K B_K) = \chi_K A_K B_K \tag{2.10}$$

2.2. Mass balances

Let ρ_k and $\underline{v}_k$ being the density and velocity fields for phase k. A so-called "single fluid" density and a "single fluid" velocity can be defined as:

$$\rho \equiv \sum_k \chi_k \rho_k , \quad \underline{v} \equiv \sum_k \chi_k \underline{v}_k \tag{2.11}$$

Due to the property (2.10), we can write:

$$\rho \underline{v} = \sum_k \chi_k \rho_k \underline{v}_k \tag{2.12}$$

When there is no net mass generation in the two-phase medium as a whole, the "single fluid" verifies the well known mass balance equation:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \underline{v}) = 0 \tag{2.13}$$

Hence, inserting (2.11) and (2.12) into (2.13):

$$\frac{\partial}{\partial t}\left(\sum_k \chi_k \rho_k \right) + \nabla.\left(\sum_k \chi_k \rho_k \underline{v}_k \right) = 0 \tag{2.14}$$

Splitting derivatives, Eq. (2.14) can be rewritten:

$$\sum_k \chi_k \left(\frac{\partial \rho_k}{\partial t} + \nabla.(\rho_k \underline{v}_k) \right) + \sum_k \rho_k \left(\frac{\partial \chi_k}{\partial t} + \underline{v}_k.\nabla \chi_k \right) = 0 \tag{2.15}$$

Inside one of the two phases (i.e. outside the interfacial surface), the derivatives of the PIF are nil, according to their expressions (2.6), and one is left with the usual mass balance equation for phase k (k = 1 or 2):

$$\frac{\partial \rho_k}{\partial t} + \nabla.(\rho_k \underline{v}_k) = 0 \tag{2.16}$$

Combining (2.16) with (2.15) gives the following equation valid on the interfaces:

$$\sum_k \rho_k \left(\underbrace{\frac{\partial \chi_k}{\partial t} + \underline{w}.\nabla \chi_k}_{0} + (\underline{v}_k - \underline{w}).\nabla \chi_k \right) = 0 \tag{2.17}$$

Taking the topological equation (2.7) into account, the first two terms disappear, and one is left with:

$$\sum_k \rho_k (\underline{v}_k - \underline{w}).\nabla \chi_k = 0 \tag{2.18}$$

which is the mass balance for the interfaces and gives the complement to the mass balance inside phases (2.16). Using (2.8) and defining:

$$\dot{m}_k \delta_I \equiv \rho_k (\underline{v}_k - \underline{w}).\nabla\chi_k = \rho_k (\underline{w} - \underline{v}_k).\underline{n}_k \delta_I \tag{2.19}$$

which is the mass *gain* due to phase change (evaporation or condensation) per unit volume, the quantity $\dot{m}_k$ being the mass gain per unit surface. The interfacial mass balance equation (2.18) is therefore:

$$\sum_k \dot{m}_k = 0 \tag{2.20}$$

showing that there is no mass accumulation at the interfaces, a consequence of the implicit assumption that the interfaces are immaterial surfaces, carrying no mass.

2.3. Momentum balances (Newton’s law)

Proceeding in the same manner as for the mass balances, the mixture momentum balance reads (Kataoka, 1986):

$$\frac{\partial}{\partial t}\left(\sum_k \chi_k \rho_k \underline{v}_k\right) + \nabla.\left(\sum_k \chi_k \rho_k \underline{v}_k \underline{v}_k\right) = -\nabla\left(\sum_k \chi_k p_k\right) + \nabla.\left(\sum_k \chi_k \underline{\underline{\tau}}_k\right) + \sum_k \chi_k \rho_k \underline{g} + \underline{F}_s \delta_I \tag{2.21}$$

where p_k and $\underline{\underline{\tau}}_k$ denote the pressure and the viscous stress tensor in phase k. The vectors $\underline{g}$ and $\underline{F}_s$ denote the gravity acceleration and the surface tension force respectively. According to Delhaye (1974), the surface tension force per unit interfacial surface has the following expression:

$$\underline{F}_s = -\sigma \underline{n} \nabla.\underline{n} + \nabla_s \sigma \tag{2.22}$$

where it is not useful to precise the sense of the unit normal vector $\underline{n}$ since it appears twice in the first term of the RHS (Right Hand Side) of (2.22). The divergence of the unit normal vector $\nabla.\underline{n}$ gives the total curvature, equal to twice the mean curvature (Aris, 1962), and the last term $\nabla_s \sigma$ is the surface gradient of the surface tension coefficient: the so-called Marangoni effect.

Defining a mixture pressure p and a mixture viscous stress tensor $\underline{\underline{\tau}}$ as in (2.11):

$$p \equiv \sum_k \chi_k p_k, \quad \underline{\underline{\tau}} \equiv \sum_k \chi_k \underline{\underline{\tau}}_k \tag{2.23}$$

It is easy to see that the momentum equation in the “single fluid” formulation reads:

$$\frac{\partial}{\partial t}(\rho \underline{v}) + \nabla.(\rho \underline{v}\underline{v}) = -\nabla p + \nabla.\underline{\underline{\tau}} + \rho \underline{g} - (\sigma \underline{n} \nabla.\underline{n} - \nabla_s \sigma)\delta_I \tag{2.24}$$

Except for the last term, the equation (2.24) is the same than the momentum balance for a single fluid, hence the name of the formulation. The last term, specific to two-phase flows, represents the surface tension force acting on a unit volume of the two-phase medium containing interfaces.

Proceeding as for the mass balances, the momentum balances for phase k and for the interface can be separated from the mixture balance (2.21). The momentum balance for phase k is the classical one:

$$\frac{\partial \rho_k \underline{v}_k}{\partial t} + \nabla.\left(\rho_k \underline{v}_k \underline{v}_k\right) = -\nabla p_k + \nabla.\underline{\underline{\tau}}_k + \rho_k \underline{g} \tag{2.25}$$

and the momentum balance for the interface reads:

$$\sum_k \dot{m}_k \underline{v}_k - p_k \underline{n}_k + \underline{\underline{\tau}}_k.\underline{n}_k = \underline{F}_s = -\sigma \underline{n} \nabla.\underline{n} + \nabla_s \sigma \tag{2.26}$$

2.4. Total energy balances (first principle)

According to Kataoka (1986), the total energy for the two-phase mixture is defined by:

$$\sum_k \chi_k \rho_k \left(e_k + \frac{{v_k}^2}{2} \right) + u_s \delta_I \tag{2.27}$$

where e_k is the specific internal energy for phase k and u_s is the interfacial energy per unit surface. The mixture total energy obeys to the following balance equation (Kataoka, 1986):

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\sum_k \chi_k \rho_k \left(e_k + \frac{{v_k}^2}{2} \right) + u_s \delta_I \right] + \nabla.\left[\sum_k \chi_k \rho_k \left(e_k + \frac{{v_k}^2}{2} \right) \underline{v}_k + u_s \delta_I \underline{w} \right] = \\ &\quad = -\nabla.\left(\sum_k \chi_k \underline{q}_k \right) - \nabla.\left(\sum_k \chi_k p_k \underline{v}_k \right) + \nabla.\left(\sum_k \chi_k \underline{\underline{\tau}}_k.\underline{v}_k \right) + \sum_k \chi_k \rho_k \underline{v}_k.\underline{g} \\ &\quad + \sum_k \chi_k Q_k + \underline{F}_s.\underline{w}\delta_I + \Gamma_s u_s \delta_I \end{aligned} \tag{2.28}$$

where $\underline{q}_k$ denotes the heat flux due to conduction inside phase k, Q_k denotes a possible heat source in phase k and Γ_s denotes a source term of δ_I. According to Kataoka (1986), the local instantaneous IAC (Interfacial Area Concentration) obeys to the following balance:

$$\frac{\partial \delta_I}{\partial t} + \nabla.\left[\delta_I \underline{w}\right] = \Gamma_s \delta_I \tag{2.29}$$

Morel (2007) gives the exact expression of the source term Γ_s due to stretching of the interfaces (i.e. in the absence of break-up and coalescence):

$$\Gamma_s = \nabla_s.\underline{w} \tag{2.30}$$

From (2.28), it is possible to separate the total energy balances for phase k and for the interface in the same manner as in the preceding paragraphs. The total energy balance equation in phase k reads:

$$\frac{\partial}{\partial t}\left[\rho_k\left(e_k+\frac{v_k{}^2}{2}\right)\right]+\nabla.\left[\rho_k\left(e_k+\frac{v_k{}^2}{2}\right)\underline{v}_k\right]=-\nabla.\underline{q}_k-\nabla.\left(p_k\underline{v}_k\right)+\nabla.\left(\underline{\underline{\tau}}_k.\underline{v}_k\right)+\rho_k\underline{v}_k.\underline{g}+Q_k \tag{2.31}$$

Using (2.29) and (2.31), the remaining of (2.28) gives the total energy balance for the interface:

$$\frac{\partial u_s}{\partial t}+\underline{w}.\nabla u_s=\underline{F}_s.\underline{w}+\sum_k\left\{\left[\underline{q}_k+p_k\underline{v}_k-\underline{\underline{\tau}}_k.\underline{v}_k\right].\underline{n}_k-\dot{m}_k\left(e_k+\frac{v_k{}^2}{2}\right)\right\} \tag{2.32}$$

Now, defining the following additional mixture quantities:

$$e\equiv\sum_k\chi_k e_k,\quad \underline{q}\equiv\sum_k\chi_k\underline{q}_k,\quad Q\equiv\sum_k\chi_k Q_k \tag{2.33}$$

The equation (2.28) can be rewritten in the “single fluid” formulation:

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\rho\left(e+\frac{v^2}{2}\right)+u_s\delta_I\right]+\nabla.\left[\rho\left(e+\frac{v^2}{2}\right)\underline{v}+u_s\delta_I\underline{w}\right]=\\&=-\nabla.\underline{q}-\nabla.\left(p\underline{v}\right)+\nabla.\left(\underline{\underline{\tau}}.\underline{v}\right)+\rho\underline{v}.\underline{g}+Q+\underline{F}_s.\underline{w}\delta_I+\Gamma_s u_s\delta_I\end{aligned} \tag{2.34}$$

2.5. Secondary balance equations

Taking the dot product of the momentum equation (2.25) by the velocity $\underline{v}_k$ gives the kinetic energy balance equation:

$$\frac{\partial\left(\rho_k\frac{v_k{}^2}{2}\right)}{\partial t}+\nabla.\left(\rho_k\frac{v_k{}^2}{2}\underline{v}_k\right)=-\nabla.\left(p_k\underline{v}_k\right)+p_k\nabla.\underline{v}_k+\nabla.\left(\underline{\underline{\tau}}_k.\underline{v}_k\right)-\underline{\underline{\tau}}_k:\underline{\underline{\nabla\underline{v}_k}}+\rho_k\underline{v}_k.\underline{g} \tag{2.35}$$

Subtracting the kinetic energy balance (2.35) from the total energy balance (2.31) gives the following internal energy balance equation:

$$\frac{\partial}{\partial t}\left[\rho_k e_k\right]+\nabla.\left[\rho_k e_k\underline{v}_k\right]=-\nabla.\underline{q}_k+Q_k-p_k\nabla.\underline{v}_k+\underline{\underline{\tau}}_k:\underline{\underline{\nabla\underline{v}_k}} \tag{2.36}$$

Substracting e_k multiplied by the mass balance equation (2.16) from (2.36), the non conservative form of (2.36) is obtained:

$$\rho_k \frac{D_k e_k}{Dt} = -\nabla.\underline{q}_k + Q_k - p_k \nabla.\underline{v}_k + \underline{\underline{\tau}}_k : \underline{\underline{\nabla \underline{v}}}_k \quad \text{with :}$$
$$\frac{D_k e_k}{Dt} \hat{=} \frac{\partial e_k}{\partial t} + \underline{v}_k.\nabla e_k \qquad (2.37)$$

The notation D_k/Dt stands for the material (or convective) derivative following the phase k in its motion. Using $\partial e_k = C_{v,k} * \partial T_k$ where $C_{v,k}$ and T_k denote respectively the specific heat at constant volume and the phase temperature, Eq. (2.37) can be rewritten for the temperature as the main variable:

$$\rho_k C_{v,k} \frac{D_k T_k}{Dt} = \nabla.(\lambda_k \nabla T_k) + Q_k - p_k \nabla.\underline{v}_k + \underline{\underline{\tau}}_k : \underline{\underline{\nabla \underline{v}}}_k \quad \text{with :}$$
$$\underline{q}_k = -\lambda_k \nabla T_k \qquad (2.38)$$

where the Fourier's law has been assumed to express the conductive heat flux $\underline{q}_k$.

Defining the enthalpy by the sum of the internal energy and the energy associated to pressure force:

$$h_k \equiv e_k + \frac{p_k}{\rho_k} \qquad (2.39)$$

The internal energy (2.36) or (2.37) can be rewritten in the form of an enthalpy balance equation:

$$\frac{\partial}{\partial t}[\rho_k h_k] + \nabla.[\rho_k h_k \underline{v}_k] = -\nabla.\underline{q}_k + Q_k + \frac{D_k p_k}{Dt} + \underline{\underline{\tau}}_k : \underline{\underline{\nabla \underline{v}}}_k \qquad (2.40)$$

Defining the total enthalpy by the sum of the enthalpy and of the kinetic energy:

$$H_k \equiv h_k + \frac{{v_k}^2}{2} \qquad (2.41)$$

The equation for H_k can be derived simply by adding the equations (2.40) and (2.35) for the two forms of energy contained in the total enthalpy:

$$\frac{\partial}{\partial t}[\rho_k H_k] + \nabla.[\rho_k H_k \underline{v}_k] = -\nabla.\underline{q}_k + Q_k + \frac{\partial p_k}{\partial t} + \nabla.\left(\underline{\underline{\tau}}_k.\underline{v}_k\right) + \rho_k \underline{v}_k.\underline{g} \qquad (2.42)$$

2.5. Two-fluid formulation

The equations (2.13), (2.24) and (2.34) constitute the "single fluid" or "one fluid" formulation for the two-phase flow. Another useful formulation is the "two-fluid" formulation where the equations for the two phases are treated (or solved) independently. These equations can be obtained from the single phase balance equations for mass (2.16), momentum (2.25) and total energy (2.31) by multiplying them by the PIF χ_k and reintroducing χ_k in the derivatives. The mass balance of the two fluid formulation reads:

$$\frac{\partial \chi_k \rho_k}{\partial t} + \nabla.\left(\chi_k \rho_k \underline{v}_k\right) = \rho_k \left(\underline{w} - \underline{v}_k\right).\underline{n}_k \delta_I \equiv \dot{m}_k \delta_I \tag{2.43}$$

The momentum balance of the two fluid formulation reads:

$$\frac{\partial \chi_k \rho_k \underline{v}_k}{\partial t} + \nabla.\left(\chi_k \rho_k \underline{v}_k \underline{v}_k\right) = \dot{m}_k \underline{v}_k \delta_I - \nabla\left(\chi_k p_k\right) - p_k \underline{n}_k \delta_I + \nabla.\left(\chi_k \underline{\underline{\tau}}_k\right) + \underline{\underline{\tau}}_k.\underline{n}_k \delta_I + \chi_k \rho_k \underline{g} \tag{2.44}$$

The total energy balance of the two fluid formulation reads:

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\chi_k \rho_k \left(e_k + \frac{v_k^{\ 2}}{2}\right)\right] + \nabla.\left[\chi_k \rho_k \left(e_k + \frac{v_k^{\ 2}}{2}\right)\underline{v}_k\right] = \\ &= \dot{m}_k \left(e_k + \frac{v_k^{\ 2}}{2}\right)\delta_I - \nabla.\left(\chi_k \underline{q}_k\right) - \nabla.\left(\chi_k p_k \underline{v}_k\right) + \nabla.\left(\chi_k \underline{\underline{\tau}}_k.\underline{v}_k\right) + \chi_k \rho_k \underline{v}_k.\underline{g} + \chi_k Q_k \\ &- \underline{q}_k.\underline{n}_k \delta_I - p_k \underline{v}_k.\underline{n}_k \delta_I + \underline{\underline{\tau}}_k.\underline{v}_k.\underline{n}_k \delta_I \end{aligned} \tag{2.45}$$

It is easy to verify that, summing the equations (2.43), (2.44) and (2.45) on the two phases (k = 1 and 2) and taking into account the interfacial balances (2.20), (2.26) and (2.32) with the definitions (2.11), (2.23) and (2.33), the balance equations of the “single fluid” formulation can be retrieved.

2.6. Example of application 1: The Rayleigh equation for a spherical vapour bubble

We consider a single vapour bubble immersed in a liquid under the following hypotheses (Delhaye, 1981):

(H1) no gravity
(H2) spherical symmetry
(H3) single component liquid
(H4) Newtonian liquid
(H5) Constant liquid viscosity μ_L
(H6) liquid obeying Fourier’s law
(H7) Constant liquid thermal conductivity λ_L
(H8) Single component vapour
(H9) Newtonian vapour
(H10) Constant vapour viscosity μ_V
(H11) vapour obeying Fourier’s law
(H12) Constant vapour thermal conductivity λ_V
(H13) Constant surface tension σ

2.6.1. Mass balance equations

Due to the assumption of problem spherical symmetry (H2), the equations are written in spherical coordinates. Under the assumptions (H3) and (H8), the mass balance equations for the liquid and vapour phases (2.16) become:

$$\begin{aligned} &\frac{\partial \rho_L}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho_L w_L\right) = 0 \\ &\frac{\partial \rho_V}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \rho_V w_V\right) = 0 \end{aligned} \qquad (2.46)$$

Where r is the radial distance to the bubble centre, of radius R, and $w_{V,L}$ are the radial component of the vapour and liquid velocities. The interfacial mass balance equation (2.20) reads:

$$\rho_{VI}\left(w_{VI} - \dot{R}\right) = \rho_{LI}\left(w_{LI} - \dot{R}\right) \quad \text{on } r = R \qquad (2.47)$$

Where $\dot{R}$ is the time rate of change of the bubble radius R and is equal to the normal displacement speed of the interface $\underline{w}.\underline{n}$ on this simple problem. The second index I indicates that the values are taken at the interface.

2.6.2. Momentum balance equations

Under the assumptions (H1), (H2), (H3), (H4), (H5), (H8), (H9) and (H10), the liquid and vapour momentum balance equations (2.25) read, in spherical coordinates:

$$\begin{aligned} &\rho_L\left(\frac{\partial w_L}{\partial t} + w_L\frac{\partial w_L}{\partial r}\right) = -\frac{\partial p_L}{\partial r} + \frac{4}{3}\mu_L\left[\frac{\partial^2 w_L}{\partial r^2} + \frac{2}{r}\frac{\partial w_L}{\partial r} - 2\frac{w_L}{r^2}\right] \\ &\rho_V\left(\frac{\partial w_V}{\partial t} + w_V\frac{\partial w_V}{\partial r}\right) = -\frac{\partial p_V}{\partial r} + \frac{4}{3}\mu_V\left[\frac{\partial^2 w_V}{\partial r^2} + \frac{2}{r}\frac{\partial w_V}{\partial r} - 2\frac{w_V}{r^2}\right] \end{aligned} \qquad (2.48)$$

The interfacial momentum balance equation (2.26) reads, under the additional assumption (H13):

$$p_{VI} - p_{LI} = \frac{2\sigma}{R} - \rho_{VI}\left(w_{VI} - \dot{R}\right)w_{VI} + \rho_{LI}\left(w_{LI} - \dot{R}\right)w_{LI} + \frac{4}{3}\mu_V\left[\frac{\partial w_V}{\partial r}\Big|_I - \frac{w_{VI}}{R}\right] - \frac{4}{3}\mu_L\left[\frac{\partial w_L}{\partial r}\Big|_I - \frac{w_{LI}}{R}\right]$$

(2.49)

2.6.3. Energy balance equations

The chosen form of the energy balance equations is the temperature equation (2.38) using assumptions (H6) and (H11). Using also (H2), (H7) and (H12), the equation (2.38) written for the two phases in spherical coordinates read:

$$\rho_L C_{v,L}\left(\frac{\partial T_L}{\partial t}+w_L\frac{\partial T_L}{\partial r}\right)=\frac{\lambda_L}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_L}{\partial r}\right)-p_L\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 w_L\right)+\underline{\underline{\tau}}_L:\underline{\underline{\nabla \underline{v}}}_L$$
$$\rho_V C_{v,V}\left(\frac{\partial T_V}{\partial t}+w_V\frac{\partial T_V}{\partial r}\right)=\frac{\lambda_V}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_V}{\partial r}\right)-p_V\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 w_V\right)+\underline{\underline{\tau}}_V:\underline{\underline{\nabla \underline{v}}}_V \tag{2.50}$$

where we made the additional assumption:

(H14): no heat source Q_k in the bulk of the phases.

The interfacial energy balance is not useful and will not be written here.

2.6.4. Rayleigh equation

In the studies of vapour bubble dynamics, a special form of the liquid momentum equation, the Rayleigh equation, is often used. We make two additional assumptions:

(H15) The liquid is incompressible
(H16) the vapour density is negligible with respect to the liquid density.

The assumption (H15) allows to integrate directly the liquid mass balance equation $(2.46)_1$ which gives:

$$w_L=\frac{A(t)}{r^2} \tag{2.51}$$

As a result, the liquid momentum equation $(2.48)_1$ takes the following simple form, whatever the liquid viscosity:

$$\rho_L\left(\frac{\partial w_L}{\partial t}+w_L\frac{\partial w_L}{\partial r}\right)=-\frac{\partial p_L}{\partial r} \tag{2.52}$$

Taking the result (2.51) into account and integrating (2.52) from R to infinity gives the following result:

$$\dot{w}_{LI}R+2\dot{R}w_{LI}-\frac{w_{LI}^{\;2}}{2}=\frac{p_{LI}-p_{L\infty}}{\rho_L} \tag{2.53}$$

where the second index I indicates quantities which are taken on the bubble interface (r = R(t)) and the overdot denotes a time derivative. Assumption (H16) enables to simplify the interfacial mass balance (2.47) as $w_{LI}\cong\dot{R}$, therefore Eq. (2.53) becomes:

$$R\ddot{R}+\frac{3}{2}\dot{R}^2=\frac{p_{LI}-p_{L\infty}}{\rho_L} \tag{2.54}$$

which is called the Rayleigh equation.

2.7. Example of application 2: Hadamard solution for the translation of a spherical inclusion in a very viscous fluid.

2.7.1. Simplifying assumptions, balance equations and boundary conditions (problem setting)

The solution presented hereafter has been derived by Hadamard (1911) and summarized by Cartellier (2008). It consists in the study of the translation of a very viscous droplet in a very viscous fluid (creeping flow). The following assumptions are necessary:

(H1) the flow is stationary
(H2) the two phases are incompressible
(H3) creeping flow, i.e. Re << 1.
(H4) Newtonian fluids with constant viscosities
(H5) the spherical droplet is translating without any acceleration
(H6) no phase change (i.e. neither evaporation nor condensation)
(H7) the flow is assumed to be axisymetric
(H8) Constant surface tension σ

Under the four assumptions (H1)-(H4), the mass and momentum balance equations in each phase (2.16) and (2.25) reduce to:

$$\begin{aligned} &\nabla.\underline{v}_k = 0 \\ &\nabla p_{m,k} = \mu_k \nabla^2 \underline{v}_k \end{aligned} \qquad (2.55)$$

Where $p_{m,k}$ is a modified pressure for phase k, including the gravity term. Introducing the vorticity $\underline{\omega}_k$ = rot($\underline{v}_k$), the second equation (2.55) can be rewritten:

$$\nabla p_{m,k} = -\mu_k \nabla \wedge \underbrace{\nabla \wedge \underline{v}_k}_{\underline{\omega}_k} = -\mu_k \nabla \wedge \underline{\omega}_k \qquad (2.56)$$

Taking successively the rotational and the divergence of (2.56), the pressure and the vorticity are harmonic quantities:

$$\begin{aligned} &\nabla \wedge \nabla \wedge \underline{\omega}_k = 0 = \nabla \underbrace{\nabla.\underline{\omega}_k}_{0} - \nabla^2 \underline{\omega}_k \quad \Rightarrow \quad \nabla^2 \underline{\omega}_k = 0 \\ &\nabla^2 p_{m,k} = 0 \end{aligned} \qquad (2.57)$$

In the following, the dispersed phase (the inclusion) will be denoted by k = 1 and the continuous fluid surrounding this inclusion will be denoted by k = 2.

Under the condition of no phase change (H6) and if we admit no slip of one phase on the other at the interface, the kinematical conditions on the spherical globule surface read:

$$\begin{aligned} &\underline{v}_1.\underline{n} = \underline{v}_2.\underline{n} \\ &\underline{v}_{t1} = \underline{v}_{t2} \end{aligned} \tag{2.58}$$

Under the assumptions (H6) and (H8), the interfacial momentum balance equation (2.26) degenerates into:

$$\sum_k \left(-p_k \underline{n}_k + \underline{\underline{\tau}}_k.\underline{n}_k\right) = -\sigma\underline{n}\nabla.\underline{n} \tag{2.59}$$

Projecting (2.59) into the normal (to the interface) and tangential directions gives:

$$\begin{aligned} &p_1 - p_2 - \tau_{1,nn} + \tau_{2,nn} = 2H\sigma \\ &\underline{\tau}_{1,nt} = \underline{\tau}_{2,nt} \end{aligned} \tag{2.60}$$

where H is the mean curvature, equal to 1/R for a spherical surface of radius R.

The boundary conditions (BC) at infinity read:

$$\begin{aligned} &\underline{v}_2 = -U\underline{e}_z \quad r \to \infty \\ &p_2 = p_0 \quad r \to \infty \end{aligned} \tag{2.61}$$

where U is the translation velocity of the globule centre. We choose to work in a reference frame linked to the globule. Let z be the symmetry axis (H7) and θ be the angle measured from the z axis.

2.7.2. Determination of the velocity and pressure fields in and around the globule

We are searching the solution under the following form:

$$v_{kr} = \frac{1}{r^2 \sin\theta}\frac{\partial\psi_k}{\partial\theta}, \qquad v_{k\theta} = -\frac{1}{r\sin\theta}\frac{\partial\psi_k}{\partial r} \tag{2.62}$$

Where ψ_k is called the Stokes stream function. The vorticity vector $\underline{\omega}_k$ has only one non zero component ω_k along the base vector $\underline{e}_\phi$ (H7) which reads :

$$\begin{aligned} &\underline{\omega}_k = \omega_k \underline{e}_\varphi \quad \text{with :} \\ &\omega_k = \frac{1}{r}\frac{\partial}{\partial r}\left(r v_{k\theta}\right) - \frac{\partial v_{kr}}{r\partial\theta} = -\frac{1}{r\sin\theta}E^2\left(\psi_k\right) \quad \text{with :} \\ &E^2 \mathrel{\hat{=}} \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot g\theta}{r^2}\frac{\partial}{\partial\theta} \end{aligned} \tag{2.63}$$

Hence, the first equation (2.57) becomes:

$$E^4\left(\psi_k\right) = 0 \tag{2.64}$$

where the operator E^4 means the operator E^2 applied two times.

The BCs (2.61), (2.58) and (2.60) become:

$$\begin{aligned}
&\frac{1}{r^2\sin\theta}\frac{\partial\psi_2}{\partial\theta}\xrightarrow[r\to\infty]{}-U\cos\theta\\
&-\frac{1}{r\sin\theta}\frac{\partial\psi_2}{\partial r}\xrightarrow[r\to\infty]{}U\sin\theta\\
&p_2=p_0\quad r\to\infty\\
&\frac{\partial\psi_1}{\partial\theta}=\frac{\partial\psi_2}{\partial\theta}=0\quad \text{en } r=R\\
&\frac{\partial\psi_1}{\partial r}=\frac{\partial\psi_2}{\partial r}\quad \text{en } r=R\\
&p_1-p_2-2\mu_1\frac{\partial}{\partial r}\left[\frac{1}{r^2\sin\theta}\frac{\partial\psi_1}{\partial\theta}\right]+2\mu_2\frac{\partial}{\partial r}\left[\frac{1}{r^2\sin\theta}\frac{\partial\psi_2}{\partial\theta}\right]=\frac{2\sigma}{R}\quad \text{en } r=R\\
&\mu_1\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi_1}{\partial r}\right)=\mu_2\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial\psi_2}{\partial r}\right)\quad \text{en } r=R
\end{aligned}\tag{2.65}$$

The solution is searched in the following form:

$$\psi_k(r,\theta)=F_k(r)\sin^2\theta \tag{2.66}$$

We obtain:

$$\begin{aligned}
&E^2(\psi_k)=\sin^2\theta\underbrace{\left(F_k''-2F_k/r^2\right)}_{f_k}\\
&E^4(\psi_k)=\sin^2\theta\left(f_k''-2f_k/r^2\right)=0\\
&\Rightarrow\quad r^4F_k^{(4)}-4r^2F_k''+8rF_k'-8F_k=0\\
&\Rightarrow\quad F_k(r)=A_kr^4+B_kr+C_kr^2+D_k/r\\
&\Rightarrow\quad v_{kr}=2\cos\theta\left(A_kr^2+\frac{B_k}{r}+C_k+\frac{D_k}{r^3}\right)\\
&\text{et}\quad v_{k\theta}=-\sin\theta\left(4A_kr^2+\frac{B_k}{r}+2C_k-\frac{D_k}{r^3}\right)
\end{aligned}\tag{2.67}$$

For the external field, the velocity remains finite when r tends to infinity, it implies $A_2 = 0$. Otherwise, the first two BCs (2.65) give $C_2 = -U/2$. For the internal field, the velocity must remain finite at the origin (i.e. $r = 0$) so we have necessarily: $B_1 = D_1 = 0$. Whence :

$$
\begin{aligned}
&\psi_1(r,\theta) = \left(A_1 r^4 + C_1 r^2\right)\sin^2\theta \\
&\psi_2(r,\theta) = \left(B_2 r - \frac{U}{2} r^2 + D_2 / r\right)\sin^2\theta \\
&v_{1r} = 2\cos\theta\left(A_1 r^2 + C_1\right) \\
&v_{1\theta} = -\sin\theta\left(4A_1 r^2 + 2C_1\right) \\
&v_{2r} = 2\cos\theta\left(\frac{B_2}{r} + \frac{D_2}{r^3} - \frac{U}{2}\right) \\
&v_{2\theta} = -\sin\theta\left(\frac{B_2}{r} - \frac{D_2}{r^3} - U\right)
\end{aligned}
\tag{2.68}
$$

The fourth line in (2.65) gives the two conditions $v_{1r} = v_{2r} = 0$ at the interface $r = R$:

$$
\begin{aligned}
&A_1 R^2 + C_1 = 0 \\
&\frac{B_2}{R} + \frac{D_2}{R^3} - \frac{U}{2} = 0
\end{aligned}
\tag{2.69}
$$

The fifth line in (2.65) gives the unique condition $v_{1\theta} = v_{2\theta}$ at the interface $r = R$:

$$
4A_1 R^2 + 2C_1 = \frac{B_2}{R} - \frac{D_2}{R^3} - U \tag{2.70}
$$

Finally, the last BC (2.65) (equality of tangential stresses at the interface) gives :

$$
\mu_1\left(4A_1 - 2\frac{C_1}{R^2}\right) = \mu_2\left(-2\frac{B_2}{R^3} + \frac{U}{R^2} + 4\frac{D_2}{R^5}\right) \tag{2.71}
$$

Introducing the following notation of the viscosities ratio:

$$
\kappa \mathrel{\hat{=}} \frac{\mu_1}{\mu_2} \tag{2.72}
$$

The solution of the system of algebraic equations (2.69)-(2.71) is:

$$
\begin{aligned}
&A_1 = -\frac{U}{4R^2}\frac{1}{1+\kappa} \\
&C_1 = \frac{U}{4}\frac{1}{1+\kappa} \\
&B_2 = \frac{UR}{2}\frac{3\kappa/2+1}{1+\kappa} \\
&D_2 = -\frac{UR^3}{4}\frac{\kappa}{1+\kappa}
\end{aligned}
\tag{2.73}
$$

The stream functions and the velocities in each phase are then deduced:

$$
\begin{aligned}
\psi_1(r,\theta) &= \left(-\frac{U}{4R^2}\frac{1}{1+\kappa}r^4+\frac{U}{4}\frac{1}{1+\kappa}r^2\right)\sin^2\theta \\
\psi_2(r,\theta) &= \left(\frac{UR}{2}\frac{3\kappa/2+1}{1+\kappa}r-\frac{U}{2}r^2-\frac{UR^3}{4r}\frac{\kappa}{1+\kappa}\right)\sin^2\theta \\
v_{1r} &= 2\cos\theta\left(-\frac{U}{4R^2}\frac{1}{1+\kappa}r^2+\frac{U}{4}\frac{1}{1+\kappa}\right) \\
v_{1\theta} &= -\sin\theta\left(-\frac{U}{R^2}\frac{1}{1+\kappa}r^2+\frac{U}{2}\frac{1}{1+\kappa}\right) \\
v_{2r} &= 2\cos\theta\left(\frac{UR}{2r}\frac{3\kappa/2+1}{1+\kappa}-\frac{UR^3}{4r^3}\frac{\kappa}{1+\kappa}-\frac{U}{2}\right) \\
v_{2\theta} &= -\sin\theta\left(\frac{UR}{2r}\frac{3\kappa/2+1}{1+\kappa}+\frac{UR^3}{4r^3}\frac{\kappa}{1+\kappa}-U\right)
\end{aligned}
\tag{2.74}
$$

The momentum equation $(2.55)_2$ then allows to calculate the modified pressure field in each phase:

$$
\begin{aligned}
p_{m,1} &= p_0' - \mu_1\frac{Ur\cos\theta}{R^2}\frac{5}{1+\kappa} \\
p_{m,2} &= p_0 + \mu_2\frac{UR\cos\theta}{r^2}\frac{2+3\kappa}{2(1+\kappa)}
\end{aligned}
\tag{2.75}
$$

The pressure value at the inclusion centre is deduced from the normal balance at the interface (see $(2.65)_6$) :

$$
p_0' = p_0 + \frac{2\sigma}{R} \tag{2.76}
$$

The surface tension σ contributes to the pressure level inside the inclusion.

2.7.3. Force exerted on the globule

The force exerted on the inclusion by the surrounding fluid is given by the following expression:

$$
\underline{F} = \int \underline{\underline{\sigma}}_2 . \underline{n}\, dS \quad \text{with} \quad \underline{\underline{\sigma}}_2 . \underline{n} = -p_2\underline{e}_r + \tau_{2,rr}\underline{e}_r + \tau_{2,r\theta}\underline{e}_\theta + \underbrace{\tau_{2,r\varphi}}_{0}\underline{e}_\varphi \tag{2.77}
$$

By symmetry , this force has only one non zero component F_z in the direction of the relative velocity :

$$
F_z = \int\left(-p_2\cos\theta + \tau_{2,rr}\cos\theta + \tau_{2,r\theta}\sin\theta\right)dS \quad \text{with} \quad dS = R^2\sin\theta d\theta d\varphi \tag{2.78}
$$

This force, often called the drag force, includes three contributions:

- Pressure force (form drag):

$$F_{z,p} = \int \left(-p_2 \cos\theta\right) dS = -2\pi\mu_2 RU \frac{2/3+\kappa}{1+\kappa} \tag{2.79}$$

- Viscous force normal to the interface (skin drag):

$$F_{z,rr} = \int \left(\tau_{2,rr} \cos\theta\right) dS = -\frac{8}{3}\pi\mu_2 RU \frac{1}{1+\kappa} \tag{2.80}$$

- Viscous force tangent to the interface (skin drag):

$$F_{z,r\theta} = \int \left(\tau_{2,r\theta} \sin\theta\right) dS = -4\pi\mu_2 RU \frac{\kappa}{1+\kappa} \tag{2.81}$$

The sum of the three contributions (2.79)-(2.81) gives the total drag force exerted on the globule:

$$F_z = -6\pi\mu_2 RU \frac{\kappa+2/3}{1+\kappa} \quad \Leftrightarrow \quad C_D = \frac{24}{Re}\frac{\kappa+2/3}{1+\kappa} \quad \text{avec} \quad Re \hat{=} \frac{2RU}{\nu_2} \tag{2.82}$$

Where C_D is the drag coefficient which is related to the drag force by the following definition:

$$C_D \equiv \frac{F_z}{\frac{1}{2}\rho_2 \pi R^2 U^2} \tag{2.83}$$

If we make the ratio κ defined by (2.72) tend to infinity (the viscosity of the fluid inclusion is infinitely greater than the one of the surrounding fluid), we retrieve the expression of the Stokes drag force on a solid particle (e.g. Oesterlé, 2006):

$$C_D = \frac{24}{Re} \qquad \text{(solid particle)} \tag{2.84}$$

The other limiting case is the one obtained by making $\kappa = 0$ (the viscosity of the fluid inclusion is nil) which corresponds approximately to the case of a clean bubble:

$$F_z = -4\pi\mu_2 RU \quad \Leftrightarrow \quad C_D = \frac{16}{Re} \quad \text{with} \quad Re \hat{=} \frac{2RU}{\nu_2} \quad \text{(clean bubble)} \tag{2.85}$$

What is interesting is to see how the detailed results (2.79)-(2.81) degenerate in these two limiting cases. The comparison of the different force contributions for the solid particle on one hand, and for the clean bubble on the other hand, are given in the following table:

Inclusion	$\kappa \hat{=} \frac{\mu_1}{\mu_2}$	$F_{z,p}$	$F_{z,rr}$	$F_{z,r\theta}$	F_z

solid	∞	$-2\pi\mu_2 RU$	0	$-4\pi\mu_2 RU$	$-6\pi\mu_2 RU$
Clean bubble	0	$-\frac{4}{3}\pi\mu_2 RU$	$-\frac{8}{3}\pi\mu_2 RU$	0	$-4\pi\mu_2 RU$

This comparison illustrates the influence of the kind of boundary conditions on the inclusion surface. It should be noted that the pressure contribution is not identical in the two cases. The tangential stresses give no contribution for a clean bubble but give the two-third of the total drag force for a solid particle. The viscous stresses in the normal direction give a contribution only for the clean bubble case, which also equals the two-third of the total drag.

3. Two-fluid averaged equations

In this section, the averaged equations of the two-fluid model are derived. These equations are the average forms of the local instantaneous two-fluid equations that have been presented in section 2.5. The expression “two-fluid” signifies that the two phases are treated separately, in opposition to some simpler models that use balance equations for the mixture considered as a whole. Nevertheless, the two phases are not independent, since they are related through interfacial interaction terms which are the average forms of the terms involving δ_I in the local instantaneous equations.

3.1. Averaging operator

Here we define the properties of the averaging operator that will be used to link the local instantaneous equations to the averaged ones. The local instantaneous equations are sometimes called “microscopic equations”, or equations at the “microscopic level”, in opposition to the averaged equations called “macroscopic” ones. The averaging operator therefore defines a bridge between the microscopic description of the flow phenomena (including all the spatial and temporal details) to a simplified macroscopic description, characterized by a smaller number of freedom degrees, which is considerably cheaper to calculate by numerical means, and which is often sufficient to the engineer. From a numerical point of view, the microscopic level is the domain of DNS (Direct Numerical Simulation) and the macroscopic one is the domain of RANS (Reynolds Averaged Navier-Stokes) simulations, to retain classical expressions used in single-phase flow calculations. An intermediate simulation domain is LES (Large Eddy Simulation) which is often used in single-phase flow calculations, but seems difficult to use for two-phase flows in the present state of the art.

There exist a lot of averaging operators used by modellers and experimentators. These averaging operators can be classified into three main categories: spatial, temporal and statistical (ensemble) averaging operators. Some composite averaging operators, resulting from the superimposition of several basic averaging operators, are sometimes used. All these operators are of different nature, each one being adapted to a certain class of problems for which they were invented. For example, the time averaging operator is more often used in the context of statistically stationary flows, even if its applicability domain is not completely restricted to this kind of flows. The more fundamental kind of averaging is the statistical averaging, which can be replaced advantageously by a time or a spatial averaging operator in some particular situations (stationary flows for time average, homogeneous flow for spatial average), invoking the ergodicity assumption. It should be kept in mind that all the averaging operators do not have the same properties. Certain averaging operators, called Reynolds operators (Sagaut, 1998) are often used because they give the simplest form of the averaged equations. In what follows, we recall the basic properties of such a Reynolds operator:

$$\langle \phi + \psi \rangle = \langle \phi \rangle + \langle \psi \rangle \tag{3.1}$$

$$\langle a\phi \rangle = a\langle \phi \rangle \qquad (a = \mathrm{cte}) \tag{3.2}$$

$$\langle \langle \phi \rangle \psi \rangle = \langle \phi \rangle \langle \psi \rangle \tag{3.3}$$

$$\left\langle \frac{\partial \phi}{\partial s} \right\rangle = \frac{\partial \langle \phi \rangle}{\partial s} \qquad s = \underline{x}, t \tag{3.4}$$

$$\langle \langle \phi \rangle \rangle = \langle \phi \rangle \tag{3.5}$$

$$\langle \phi' \rangle = 0 \qquad \phi' \mathrel{\hat{=}} \phi - \langle \phi \rangle \tag{3.6}$$

where ϕ and ψ denote two field quantities (i.e. quantities depending on time and space coordinates). The averaging operator is denoted by brackets < >. The relations (3.1) and (3.2) express the linearity of the averaging operator, a property which is common to all kinds of averaging operator. The relation (3.4) expresses that time and spatial derivatives can be permuted with the averaging operator without introducing any additional term. In two-phase flows, where discontinuous fields are encountered at the interfaces, this property is valid only by interpreting the fields in the sense of generalized functions (or distributions) which has been done in section 2. The other three relations (3.3), (3.5) and (3.6) express that an already averaged quantity is unaffected by a second application of the averaging operator. As a consequence, if the fluctuating quantity is defined as the difference between the quantity itself and its average (Eq. 3.6), the averaging of this fluctuating quantity gives zero. This very important property is specific to a Reynolds averaging operator, and is not true for example, in the case of a spatial averaging operator used in LES. The ensemble (or statistical) averaging operator has this property, and we will assume in our subsequent developments, that the chosen averaging operator is a Reynolds one.

3.2. Primary averaged balance equations

By ‘primary’, we mean the mass, momentum and total energy balance equations. Their local instantaneous form has been summarized in section 2.5, in the form of the two-fluid formulation. Hence, it is sufficient here to apply directly the averaging operator to obtain the primary averaged balance equations.

3.2.1. Mass balance equation

The average of the mass balance equation (2.43) gives:

$$\frac{\partial\langle\chi_k\rho_k\rangle}{\partial t}+\nabla.\langle\chi_k\rho_k\underline{v}_k\rangle=\langle\dot{m}_k\delta_I\rangle \tag{3.7}$$

Several macroscopic (averaged) quantities are then defined. The averaged fraction of presence of phase k, sometimes called the “void fraction” or the “phase holdup”, is defined as the average of the PIF, hence:

$$\alpha_k\equiv\langle\chi_k\rangle \tag{3.8}$$

The (intrinsic) phase averaged density is then defined by:

$$\overline{\psi_k}^k\equiv\frac{\langle\chi_k\psi_k\rangle}{\langle\chi_k\rangle}=\frac{\langle\chi_k\psi_k\rangle}{\alpha_k}\quad\text{e.g.}\quad\overline{\rho_k}^k\equiv\frac{\langle\chi_k\rho_k\rangle}{\langle\chi_k\rangle}=\frac{\langle\chi_k\rho_k\rangle}{\alpha_k} \tag{3.9}$$

Favre averaging is classically defined for terms or quantities weighted by mass or volumetric mass:

$$\overline{\overline{\phi_k}}^k\equiv\frac{\langle\chi_k\rho_k\phi_k\rangle}{\langle\chi_k\rho_k\rangle}=\frac{\langle\chi_k\rho_k\phi_k\rangle}{\alpha_k\overline{\rho_k}^k}\quad\text{e.g.}\quad\overline{\overline{\underline{v}_k}}^k\equiv\frac{\langle\chi_k\rho_k\underline{v}_k\rangle}{\langle\chi_k\rho_k\rangle}=\frac{\langle\chi_k\rho_k\underline{v}_k\rangle}{\alpha_k\overline{\rho_k}^k} \tag{3.10}$$

At the end, we also define the average mass gain per unit volume per unit time due to phase change by:

$$\Gamma_k\equiv\langle\dot{m}_k\delta_I\rangle \tag{3.11}$$

As a consequence of the preceding definitions, Eq. (3.7) can equivalently be rewritten as:

$$\frac{\partial\alpha_k\overline{\rho_k}^k}{\partial t}+\nabla.\left(\alpha_k\overline{\rho_k}^k\overline{\overline{\underline{v}_k}}^k\right)=\Gamma_k \tag{3.12}$$

The equation (3.12) is the classical form for the mass balance equation of the two-fluid model (e.g. Ishii, 1975).

Eq. (3.12) should be supplemented by the averaged form of the interfacial mass balance (2.20). Multiplying (2.20) by δ_I and taking the average, the following *mass jump condition* is obtained:

$$\sum_k\Gamma_k=0 \tag{3.13}$$

3.2.2. Momentum balance equation

Averaging Eq. (2.44), we obtain:

$$\begin{aligned}&\frac{\partial\left\langle\chi_k\rho_k\underline{v}_k\right\rangle}{\partial t}+\nabla.\left\langle\chi_k\rho_k\underline{v}_k\underline{v}_k\right\rangle=-\nabla\left\langle\chi_k p_k\right\rangle+\nabla.\left\langle\chi_k\underline{\underline{\tau}}_k\right\rangle+\alpha_k\overline{\rho_k}^k\underline{g}\\&\quad+\left\langle\dot{m}_k\underline{v}_k\delta_I\right\rangle-\left\langle p_k\underline{n}_k\delta_I\right\rangle+\left\langle\underline{\underline{\tau}}_k.\underline{n}_k\delta_I\right\rangle\end{aligned}\tag{3.14}$$

where the second line regroups the momentum interfacial transfers. The first interfacial transfer $\left\langle\dot{m}_k\underline{v}_k\delta_I\right\rangle$ is the exchange of momentum associated to the mass transfer by phase change. It is often called the “recoil force”. The two other interfacial terms are the mean interfacial forces due to pressure and viscous stresses.

Eq. (3.14) is an exact equation and reflects directly the local instantaneous form developed in section 2.5. If the mass balance equation (3.12) seems to be common to all two-fluid versions developed by the different authors, the situation is not so simple for the momentum equation. Many variants can be developed from Eq. (3.14) according to the choices made by the different authors to define macroscopic variables. These macroscopic variables should be “full of physical significance” and this physical significance depends on the problem being studied. For example, stratified flows are completely different from dispersed flows (like bubbles, droplets or particulate flows). A stratified flow is a flow where the gas and liquid phases are superimposed continua, separated by a unique continuous interface. In a stratified flow, the two phases play therefore a symmetric role and the equations should reflect this symmetry. On the contrary, for bubbly or droplet flows, one phase (called the dispersed phase) is constituted from small inclusions embedded in the other phase (called the continuous phase). The two phases are therefore not symmetric, and this asymmetry is also reflected by some two-fluid models especially devoted to dispersed flows (e.g. Zhang & Prosperetti, 1994).

In this section, we make no particular assumption on the flow configuration, and we present some definitions and manipulations which are proposed in quite general two-phase flow text books (Ishii, 1975; Drew & Passmann, 1999; Oesterlé, 2006).

Defining phase averaged pressure and viscous stress tensor in a similar manner than Eq. (3.9), Eq. (3.14) can be rewritten:

$$\begin{aligned}&\frac{\partial\alpha_k\overline{\rho_k}^k\overline{\overline{\underline{v}_k}}^k}{\partial t}+\nabla.\left(\alpha_k\overline{\rho_k}^k\overline{\overline{\underline{v}_k\underline{v}_k}}^k\right)=-\nabla\left(\alpha_k\overline{p_k}^k\right)+\nabla.\left(\alpha_k\overline{\underline{\underline{\tau}}_k}^k\right)+\alpha_k\overline{\rho_k}^k\underline{g}\\&\quad+\left\langle\dot{m}_k\underline{v}_k\delta_I\right\rangle-\left\langle p_k\underline{n}_k\delta_I\right\rangle+\left\langle\underline{\underline{\tau}}_k.\underline{n}_k\delta_I\right\rangle\end{aligned}\tag{3.15}$$

Defining the fluctuating velocity around its Favre average:

$$\underline{v}'_k\equiv\underline{v}_k-\overline{\overline{\underline{v}_k}}^k\tag{3.16}$$

It can be shown by using Eqs. (3.3), (3.5) and (3.6) that:

$$\overline{\overline{\underline{v}_k\underline{v}_k}}^k=\overline{\overline{\underline{v}_k}}^k\,\overline{\overline{\underline{v}_k}}^k+\overline{\overline{\underline{v}'_k\underline{v}'_k}}^k\tag{3.17}$$

The *Reynolds stress tensor* for phase k is defined in a manner analogous to single-phase flow by:

$$\underline{\underline{\tau}}_k^T \equiv -\overline{\rho_k}^k \, \overline{\overline{\underline{v}'_k \underline{v}'_k}}^k \tag{3.18}$$

Now we must examine the interfacial transfer terms. The recoil force is often expressed by introducing a new averaged velocity weighted by phase change:

$$\langle \dot{m}_k \underline{v}_k \delta_I \rangle \equiv \Gamma_k \underline{v}_k^\Gamma \tag{3.19}$$

The interfacial pressure force is purely normal to the interface. This is not the case of the viscous force which has normal and tangential components according to:

$$\underline{\underline{\tau}}_k . \underline{n}_k = \underbrace{\left(\underline{n}_k . \underline{\underline{\tau}}_k . \underline{n}_k\right)}_{\tau_k^{nn}} \underline{n}_k + \underline{\tau}_k^t = \underline{\tau}_k^n + \underline{\tau}_k^t \tag{3.20}$$

A first form of the averaged momentum equation can be given:

$$\frac{\partial \alpha_k \overline{\rho_k}^k \, \overline{\overline{\underline{v}_k}}^k}{\partial t} + \nabla . \left(\alpha_k \overline{\rho_k}^k \, \overline{\overline{\underline{v}_k}}^k \, \overline{\overline{\underline{v}_k}}^k \right) = -\nabla\left(\alpha_k \overline{p_k}^k\right) + \nabla . \left(\alpha_k \left(\overline{\underline{\underline{\tau}}_k}^k + \underline{\underline{\tau}}_k^T\right)\right) + \alpha_k \overline{\rho_k}^k \underline{g} + \underline{M}_k \tag{3.21}$$

where $\underline{M}_k$ is defined as the averaged interfacial momentum transfer:

$$\underline{M}_k \equiv \langle \dot{m}_k \underline{v}_k \delta_I \rangle - \langle p_k \underline{n}_k \delta_I \rangle + \langle \underline{\underline{\tau}}_k . \underline{n}_k \delta_I \rangle \tag{3.22}$$

Now we will follow the book from Ishii & Hibiki (2006) to decompose the interfacial transfer term of momentum $\underline{M}_k$. They also define the mixture momentum source due to surface tension:

$$\underline{M}_m \equiv \langle \underline{F}_s \delta_I \rangle = \langle \left(-\sigma \underline{n} \nabla . \underline{n} + \nabla_s \sigma\right) \delta_I \rangle \tag{3.23}$$

The authors introduce the surface mean values:

$$\overline{\psi}^I \equiv \frac{\langle \psi \delta_I \rangle}{\langle \delta_I \rangle} \equiv \frac{\langle \psi \delta_I \rangle}{a_I} \tag{3.24}$$

for an arbitrary function ψ defined at the interface, where a_I is defined as the averaged interfacial area concentration. Defining interfacial-averaged pressure and viscous stresses, the interfacial transfer of momentum is decomposed in the following manner:

$$\begin{aligned}
&\underline{M}_k = \underline{M}_k^{\Gamma} + \underline{M}_k^{n} + \overline{p_k}^{I} \nabla \alpha_k + \underline{M}_k^{t} - \overline{\underline{\underline{\tau}}_k}^{I} . \nabla \alpha_k \quad \text{with :} \\
&\underline{M}_k^{\Gamma} \equiv \Gamma_k \underline{v}_k^{\Gamma} \\
&\underline{M}_k^{n} \equiv \left\langle \left(\overline{p_k}^{I} - p_k \right) \underline{n}_k \delta_I \right\rangle \\
&\underline{M}_k^{t} \equiv \left\langle \left(\underline{\underline{\tau}}_k - \overline{\underline{\underline{\tau}}_k}^{I} \right) \underline{n}_k \delta_I \right\rangle
\end{aligned} \tag{3.25}$$

The first three terms in the RHS of Eq. $(3.25)_1$ are the normal components whereas the last two terms are tangential components. The void fraction gradient appears due to the averaged form of the relation $(2.8)_2$:

$$\nabla \alpha_k = -\left\langle \underline{n}_k \delta_I \right\rangle \tag{3.26}$$

The normal force $\underline{M}_k^{n}$ represents the form drag and lift force arising from the pressure imbalance at the interfaces. The tangential force $\underline{M}_k^{t}$ represents the skin drag due to the imbalance of shear forces. The two forces are combined to define the total generalized drag force:

$$\underline{M}_{ik} \equiv \underline{M}_k^{n} + \underline{M}_k^{t} \tag{3.27}$$

Introducing the averaged mean curvature of the interfaces $\overline{H_{21}}^{I}$ as well as the averaged surface tension $\overline{\sigma}^{I}$, Ishii & Hibiki (2006) rewrite the mixture momentum source in the following manner:

$$\underline{M}_m = 2\overline{H_{21}}^{I}\,\overline{\sigma}^{I} \nabla \alpha_2 + \left\langle \left(2\left(H_{21}\sigma - \overline{H_{21}}^{I}\,\overline{\sigma}^{I} \right) \underline{n}_1 + \nabla_s \sigma \right) \delta_I \right\rangle \tag{3.28}$$

The second term takes into account the effect of changes of the mean curvature. Neglecting the Marangoni effect (last term in the RHS of Eq. (3.28)), the mixture momentum source is approximated by:

$$\begin{aligned}
&\underline{M}_m \cong 2\overline{H_{21}}^{I}\,\overline{\sigma}^{I} \nabla \alpha_2 + \underline{M}_m^{H} \quad \text{with :} \\
&\underline{M}_m^{H} \equiv \left\langle 2\left(H_{21}\sigma - \overline{H_{21}}^{I}\,\overline{\sigma}^{I} \right) \underline{n}_1 \delta_I \right\rangle
\end{aligned} \tag{3.29}$$

The vector $\underline{M}_m^{H}$ is the effect of the changes of the mean curvature on the mixture momentum source.

The averaged form of the local instantaneous interfacial balance of momentum (2.26) is:

$$\sum_k \underline{M}_k = \underline{M}_m \tag{3.30}$$

Introducing the decomposition (3.25) with (3.27) into (3.21), the following form of the momentum balance is obtained:

$$\frac{\partial \alpha_k \overline{\rho_k}^{k} \overline{\overline{\underline{v}_k}}^{k}}{\partial t} + \nabla.\left(\alpha_k \overline{\rho_k}^{k} \overline{\overline{\underline{v}_k}}^{k} \overline{\overline{\underline{v}_k}}^{k}\right) = -\nabla\left(\alpha_k \overline{p_k}^{k}\right) + \nabla.\left(\alpha_k\left(\overline{\underline{\underline{\tau}}_k}^{k} + \underline{\underline{\tau}}_k^{T}\right)\right) + \alpha_k \overline{\rho_k}^{k}\underline{g} + \Gamma_k \underline{v}_k^{\Gamma} + \underline{M}_{ik} + \overline{p_k}^{I}\nabla\alpha_k - \overline{\underline{\underline{\tau}}_k}^{I}.\nabla\alpha_k \tag{3.31}$$

Subtracting from (3.31) the mass balance equation (3.12) previously multiplied by the mean velocity $\overline{\overline{\underline{v}_k}}^{k}$ and rearranging, the following non conservative form of the momentum balance equation is obtained:

$$\alpha_k \overline{\rho_k}^{k} \frac{D_k \overline{\overline{\underline{v}_k}}^{k}}{Dt} = -\alpha_k \nabla \overline{p_k}^{k} + \nabla.\left(\alpha_k\left(\overline{\underline{\underline{\tau}}_k}^{k} + \underline{\underline{\tau}}_k^{T}\right)\right) + \alpha_k \overline{\rho_k}^{k}\underline{g} + \Gamma_k\left(\underline{v}_k^{\Gamma} - \overline{\overline{\underline{v}_k}}^{k}\right) + \underline{M}_{ik} + \left(\overline{p_k}^{I} - \overline{p_k}^{k}\right)\nabla\alpha_k - \overline{\underline{\underline{\tau}}_k}^{I}.\nabla\alpha_k \tag{3.32}$$

where the material derivative following phase k in its mean motion:

$$\frac{D_k}{Dt} \equiv \frac{\partial}{\partial t} + \overline{\overline{\underline{v}_k}}^{k}.\nabla \tag{3.33}$$

3.2.3. Total energy balance equation

Averaging Eq. (2.45) gives:

$$\begin{aligned}&\frac{\partial}{\partial t}\left\langle \chi_k \rho_k\left(e_k + \frac{v_k^{2}}{2}\right)\right\rangle + \nabla.\left\langle \chi_k \rho_k\left(e_k + \frac{v_k^{2}}{2}\right)\underline{v}_k\right\rangle = \\ &= \left\langle \dot{m}_k\left(e_k + \frac{v_k^{2}}{2}\right)\delta_I\right\rangle - \nabla.\left\langle \chi_k \underline{q}_k\right\rangle - \nabla.\left\langle \chi_k p_k \underline{v}_k\right\rangle + \nabla.\left\langle \chi_k \underline{\underline{\tau}}_k.\underline{v}_k\right\rangle + \alpha_k \overline{\rho_k}^{k} \overline{\overline{\underline{v}_k}}^{k}.\underline{g} + \alpha_k \overline{Q_k}^{k} \\ &- \left\langle \underline{q}_k.\underline{n}_k \delta_I\right\rangle - \left\langle p_k \underline{v}_k.\underline{n}_k \delta_I\right\rangle + \left\langle \underline{\underline{\tau}}_k.\underline{v}_k.\underline{n}_k \delta_I\right\rangle\end{aligned} \tag{3.34}$$

Now let us define the following macroscopic quantities (Ishii & Hibiki, 2006):

- The averaged *turbulent kinetic energy*:

$$K_k \equiv \frac{\overline{\overline{v_k^{2}}}^{k}}{2} - \frac{\overline{\overline{v_k}}^{k\,2}}{2} \equiv \frac{\overline{\overline{v_k'^{2}}}^{k}}{2} \tag{3.35}$$

- The *apparent internal energy* which is the sum of the mean internal energy and the turbulent kinetic energy:

$$U_k \equiv \overline{\overline{e_k}}^{k} + K_k \tag{3.36}$$

- The *turbulent heat flux* which takes into account the turbulent energy convection as well as the turbulent work:

$$\underline{q}_k^{T} \equiv \overline{\rho_k\left(e_k + \frac{v_k^{2}}{2}\right)'\underline{v}'_k}^{k} - \overline{\underline{\underline{\tau}}_k.\underline{v}'_k}^{k} + \overline{p_k\underline{v}'_k}^{k} \tag{3.37}$$

- The *interfacial supply of total energy* to the k^th^ phase, which groups all the interfacial transfer terms in Eq. (3.34):

$$E_k \equiv \left\langle \dot{m}_k\left(e_k + \frac{v_k^{2}}{2}\right)\delta_I\right\rangle - \left\langle p_k\underline{v}_k.\underline{n}_k\delta_I\right\rangle + \left\langle \underline{\underline{\tau}}_k.\underline{v}_k.\underline{n}_k\delta_I\right\rangle - \left\langle \underline{q}_k.\underline{n}_k\delta_I\right\rangle \tag{3.38}$$

Using these definitions, Eq. (3.34) becomes:

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\alpha_k\overline{\rho_k}^{k}\left(U_k + \frac{\overline{\overline{v_k}}^{k\,2}}{2}\right)\right] + \nabla.\left[\alpha_k\overline{\rho_k}^{k}\left(U_k + \frac{\overline{\overline{v_k}}^{k\,2}}{2}\right)\overline{\overline{\underline{v}_k}}^{k}\right] = \\ &= -\nabla.\left[\alpha_k\left(\overline{\underline{q}_k}^{k} + \underline{q}_k^{T}\right)\right] - \nabla.\left(\alpha_k\overline{p_k}^{k}\,\overline{\overline{\underline{v}_k}}^{k}\right) + \nabla.\left(\alpha_k\overline{\underline{\underline{\tau}}_k}^{k}.\overline{\overline{\underline{v}_k}}^{k}\right) + \alpha_k\overline{\rho_k}^{k}\,\overline{\overline{\underline{v}_k}}^{k}.\underline{g} + \alpha_k\overline{Q_k}^{k} \\ &+ E_k\end{aligned} \tag{3.39}$$

Multiplying Eq. (2.32) by δ_I and averaging, the following jump condition for the total energy is obtained:

$$\sum_k E_k = E_s \quad \text{with} \quad E_s \equiv \left\langle\left(\underline{F}_s.\underline{w} - \frac{\partial u_s}{\partial t} - \underline{w}.\nabla u_s\right)\delta_I\right\rangle \tag{3.40}$$

where E_s is the surface energy source for the mixture. This means that energy can be stored at or released from interfaces.

Now, following the standard method of section 2.5, we will derive secondary forms of the energy balance equation. Dotting the momentum equation (3.21) by the mean velocity, the following mechanical energy equation is obtained:

$$\frac{\partial}{\partial t}\left(\alpha_k \overline{\rho_k}^{k}\frac{\overline{\overline{v_k}}^{k\,2}}{2}\right)+\nabla.\left(\alpha_k \overline{\rho_k}^{k}\frac{\overline{\overline{v_k}}^{k\,2}}{2}\overline{\overline{\underline{v}_k}}^{k}\right)=-\overline{\overline{\underline{v}_k}}^{k}.\nabla\left(\alpha_k \overline{p_k}^{k}\right)+\overline{\overline{\underline{v}_k}}^{k}.\nabla.\left(\alpha_k\left(\overline{\underline{\underline{\tau}}_k}^{k}+\underline{\underline{\tau}}_k^{T}\right)\right) \tag{3.41}$$

$$+\alpha_k \overline{\rho_k}^{k}\overline{\overline{\underline{v}_k}}^{k}.\underline{g}+\underline{M}_k.\overline{\overline{\underline{v}_k}}^{k}-\Gamma_k\frac{\overline{\overline{v_k}}^{k\,2}}{2}$$

Subtracting Eq. (3.41) from Eq. (3.39), the mean *apparent* internal energy equation is obtained:

$$\frac{\partial}{\partial t}\left[\alpha_k \overline{\rho_k}^{k} U_k\right]+\nabla.\left[\alpha_k \overline{\rho_k}^{k} U_k \overline{\overline{\underline{v}_k}}^{k}\right]=$$

$$=-\nabla.\left[\alpha_k\left(\overline{\underline{q}_k}^{k}+\underline{q}_k^{T}+\underline{\underline{\tau}}_k^{T}.\overline{\overline{\underline{v}_k}}^{k}\right)\right]-\alpha_k \overline{p_k}^{k}\nabla.\overline{\overline{\underline{v}_k}}^{k}+\alpha_k\left(\overline{\underline{\underline{\tau}}_k}^{k}+\underline{\underline{\tau}}_k^{T}\right):\underline{\underline{\nabla \overline{\overline{v_k}}^{k}}}+\alpha_k\overline{Q_k}^{k} \tag{3.42}$$

$$+E_k-\underline{M}_k.\overline{\overline{\underline{v}_k}}^{k}+\Gamma_k\frac{\overline{\overline{v_k}}^{k\,2}}{2}$$

The interfacial transfer in the thermal energy equation (3.42) has a special form which combines the mass, momentum and total energy transfer terms:

$$\Lambda_k \equiv E_k-\underline{M}_k.\overline{\overline{\underline{v}_k}}^{k}+\Gamma_k\frac{\overline{\overline{v_k}}^{k\,2}}{2} \tag{3.43}$$

Now, introducing the *apparent enthalpy* as the sum of the mean enthalpy and the turbulent kinetic energy:

$$H_k \equiv \overline{\overline{h_k}}^{k}+K_k=U_k+\frac{\overline{p_k}^{k}}{\overline{\rho_k}^{k}} \tag{3.44}$$

The thermal energy equation (3.42) can be rewritten for the apparent enthalpy:

$$\frac{\partial}{\partial t}\left[\alpha_k \overline{\rho_k}^{k} H_k\right]+\nabla.\left[\alpha_k \overline{\rho_k}^{k} H_k \overline{\overline{\underline{v}_k}}^{k}\right]=\frac{D_k\left(\alpha_k \overline{p_k}^{k}\right)}{Dt}$$

$$=-\nabla.\left[\alpha_k\left(\overline{\underline{q}_k}^{k}+\underline{q}_k^{T}+\underline{\underline{\tau}}_k^{T}.\overline{\overline{\underline{v}_k}}^{k}\right)\right]+\alpha_k\left(\overline{\underline{\underline{\tau}}_k}^{k}+\underline{\underline{\tau}}_k^{T}\right):\underline{\underline{\nabla \overline{\overline{v_k}}^{k}}}+\alpha_k\overline{Q_k}^{k}+\Lambda_k \tag{3.45}$$

Now we will examine in details the content of the interfacial thermal energy transfer Λ_k. Starting from the definitions (3.11), (3.22) and (3.38), we obtain the expression of Λ_k as a function of the microscopic fields:

$$\Lambda_k = \left\langle \dot{m}_k \left(e_k + \underbrace{\frac{v_k^{\,2}}{2} - \underline{v}_k . \overline{\overline{\underline{v}}}_k^{\,k} + \frac{\overline{\overline{v}}_k^{\,k\,2}}{2}}_{\frac{\left|\underline{v}_k - \overline{\overline{\underline{v}}}_k^{\,k}\right|^2}{2} = \frac{v_k'^{\,2}}{2}} \right) \delta_I \right\rangle + \left\langle \left(\underline{\underline{\tau}}_k . \underline{n}_k - p_k \underline{n}_k \right) . \left(\underline{v}_k - \overline{\overline{\underline{v}}}_k^{\,k} \right) \delta_I \right\rangle - \left\langle \underline{q}_k . \underline{n}_k \delta_I \right\rangle \tag{3.46}$$

Then, introducing the following macroscopic quantities:

$$\begin{aligned} \Gamma_k U_k^\Gamma &\equiv \left\langle \dot{m}_k \left(e_k + \frac{v_k'^{\,2}}{2} \right) \delta_I \right\rangle \\ q''_{kI} a_I &\equiv -\left\langle \underline{q}_k . \underline{n}_k \delta_I \right\rangle \end{aligned} \tag{3.47}$$

where U_k^Γ is the interfacial average of the apparent internal energy weighted by phase change and q''_{kI} is the heat input per unit interfacial area, a_I being the interfacial area per unit volume. Introducing the definitions (3.47) into (3.46), this becomes:

$$\Lambda_k = \Gamma_k U_k^\Gamma + q''_{kI} a_I + \left\langle \left(\underline{\underline{\tau}}_k . \underline{n}_k - p_k \underline{n}_k \right) . \left(\underline{v}_k - \overline{\overline{\underline{v}}}_k^{\,k} \right) \delta_I \right\rangle \tag{3.48}$$

The mechanical term in Eq. (3.48) is a little bit more difficult to express in terms of macroscopic quantities. The pressure and viscous stress tensor must be decomposed into interfacial average values (Eq. 3.24) and fluctuating parts. At the end, the following result is obtained (Ishii & Hibiki, 2006):

$$\left\langle \left(\underline{\underline{\tau}}_k . \underline{n}_k - p_k \underline{n}_k \right) . \left(\underline{v}_k - \overline{\overline{\underline{v}}}_k^{\,k} \right) \delta_I \right\rangle = \overline{p}_k^{\,I} \left(\frac{\Gamma_k}{\overline{\rho}_k^{\,I}} - \frac{D_k \alpha_k}{Dt} \right) + \left(\underline{M}_{ik} - \overline{\underline{\underline{\tau}}}_k^{\,I} . \nabla \alpha_k \right) . \left(\underline{v}_k^\Gamma - \overline{\overline{\underline{v}}}_k^{\,k} \right) + W_{ki}^T \tag{3.49}$$

where W_{ki}^T is the turbulent work of the interfacial forces. Substituting Eq. (3.49) into Eq. (3.48), the macroscopic interfacial thermal energy transfer becomes:

$$\Lambda_k = \Gamma_k U_k^\Gamma + q''_{kI} a_I + \overline{p}_k^{\,I} \left(\frac{\Gamma_k}{\overline{\rho}_k^{\,I}} - \frac{D_k \alpha_k}{Dt} \right) + \left(\underline{M}_{ik} - \overline{\underline{\underline{\tau}}}_k^{\,I} . \nabla \alpha_k \right) . \left(\underline{v}_k^\Gamma - \overline{\overline{\underline{v}}}_k^{\,k} \right) + W_{ki}^T \tag{3.50}$$

In analogy with (3.44), the apparent mean enthalpy at interfaces weighted by phase change can be defined:

$$H_k^\Gamma \equiv U_k^\Gamma + \frac{\overline{p}_k^{\,I}}{\overline{\rho}_k^{\,I}} \tag{3.51}$$

Then we have:

$$\Lambda_k = \Gamma_k H_k^{\Gamma} + q''_{kI} a_I - \overline{p_k}^{I} \frac{D_k \alpha_k}{Dt} + \left(\underline{M}_{ik} - \overline{\underline{\underline{\tau}}_k}^{I} . \nabla \alpha_k \right) . \left(\underline{v}_k^{\Gamma} - \overline{\overline{\underline{v}_k}}^{k} \right) + W_{ki}^{T} \tag{3.52}$$

It is straightforward to obtain E$_k$ from the relations for Λ_k, $\underline{M}_k$ and Γ_k, therefore we have from (3.25), (3.27) and (3.52) the following result:

$$E_k = \Gamma_k \left(H_k^{\Gamma} + \underline{v}_k^{\Gamma} . \overline{\overline{\underline{v}_k}}^{k} - \frac{\overline{\overline{v_k}}^{k\,2}}{2} \right) + + q''_{kI} a_I - \overline{p_k}^{I} \frac{\partial \alpha_k}{\partial t} + \left(\underline{M}_{ik} - \overline{\underline{\underline{\tau}}_k}^{I} . \nabla \alpha_k \right) . \underline{v}_k^{\Gamma} + W_{ki}^{T} \tag{3.53}$$

Substituting these results into the total energy equation (3.39) and into the apparent thermal energy equation (3.45), these equations become:

- Total energy equation:

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\alpha_k \overline{\rho_k}^{k} \left(U_k + \frac{\overline{\overline{v_k}}^{k\,2}}{2} \right) \right] + \nabla . \left[\alpha_k \overline{\rho_k}^{k} \left(U_k + \frac{\overline{\overline{v_k}}^{k\,2}}{2} \right) \overline{\overline{\underline{v}_k}}^{k} \right] = \\ &= -\nabla . \left[\alpha_k \left(\overline{\underline{q}_k}^{k} + \underline{q}_k^{T} \right) \right] - \nabla . \left(\alpha_k \overline{p_k}^{k} \overline{\overline{\underline{v}_k}}^{k} \right) + \nabla . \left(\alpha_k \overline{\underline{\underline{\tau}}_k}^{k} . \overline{\overline{\underline{v}_k}}^{k} \right) + \alpha_k \overline{\rho_k}^{k} \overline{\overline{\underline{v}_k}}^{k} . \underline{g} + \alpha_k \overline{Q_k}^{k} \\ &+ \Gamma_k \left(H_k^{\Gamma} + \underline{v}_k^{\Gamma} . \overline{\overline{\underline{v}_k}}^{k} - \frac{\overline{\overline{v_k}}^{k\,2}}{2} \right) + q''_{kI} a_I - \overline{p_k}^{I} \frac{\partial \alpha_k}{\partial t} + \left(\underline{M}_{ik} - \overline{\underline{\underline{\tau}}_k}^{I} . \nabla \alpha_k \right) . \underline{v}_k^{\Gamma} + W_{ki}^{T} \end{aligned} \tag{3.54}$$

- Apparent thermal energy equation:

$$\begin{aligned} &\frac{\partial}{\partial t}\left[\alpha_k \overline{\rho_k}^{k} H_k \right] + \nabla . \left[\alpha_k \overline{\rho_k}^{k} H_k \overline{\overline{\underline{v}_k}}^{k} \right] = \frac{D_k \left(\alpha_k \overline{p_k}^{k} \right)}{Dt} \\ &\quad = -\nabla . \left[\alpha_k \left(\overline{\underline{q}_k}^{k} + \underline{q}_k^{T} \right) \right] - \overline{\overline{\underline{v}_k}}^{k} . \nabla . \left(\alpha_k \underline{\underline{\tau}}_k^{T} \right) + \alpha_k \overline{\underline{\underline{\tau}}_k}^{k} : \underline{\underline{\nabla}} \overline{\overline{\underline{v}_k}}^{k} + \alpha_k \overline{Q_k}^{k} + \\ &\quad + \Gamma_k H_k^{\Gamma} + q''_{kI} a_I - \overline{p_k}^{I} \frac{D_k \alpha_k}{Dt} + \left(\underline{M}_{ik} - \overline{\underline{\underline{\tau}}_k}^{I} . \nabla \alpha_k \right) . \left(\underline{v}_k^{\Gamma} - \overline{\overline{\underline{v}_k}}^{k} \right) + W_{ki}^{T} \end{aligned} \tag{3.55}$$

Defining the turbulent energy source by Φ_k^{T} and the viscous dissipation term due to the mean velocity gradient by Φ_k^{D}, thus:

$$\begin{aligned} \Phi_k^{T} &\equiv -\overline{\overline{\underline{v}_k}}^{k} . \nabla . \left(\alpha_k \underline{\underline{\tau}}_k^{T} \right) + W_{ki}^{T} \\ \Phi_k^{D} &\equiv \alpha_k \overline{\underline{\underline{\tau}}_k}^{k} : \underline{\underline{\nabla}} \overline{\overline{\underline{v}_k}}^{k} \end{aligned} \tag{3.56}$$

The equation (3.55) can be rewritten equivalently into the following non conservative form:

$$\begin{aligned} \alpha_k \overline{\rho_k}^{k} \frac{D_k H_k}{Dt} &= -\nabla.\left[\alpha_k \left(\overline{\underline{q}_k}^{k} + \underline{q}_k^{T}\right)\right] + \alpha_k \frac{D_k \overline{p_k}^{k}}{Dt} + \Phi_k^{T} + \Phi_k^{D} + \alpha_k \overline{Q_k}^{k} + \\ &+ \Gamma_k \left(H_k^{\Gamma} - H_k\right) + q''_{kI} a_I + \left(\overline{p_k}^{k} - \overline{p_k}^{I}\right) \frac{D_k \alpha_k}{Dt} + \left(\underline{M}_{ik} - \overline{\underline{\underline{\tau}}_k}^{I}.\nabla\alpha_k\right).\left(\underline{v}_k^{\Gamma} - \overline{\overline{\underline{v}_k}}^{k}\right) \end{aligned} \tag{3.57}$$

3.3. Averaged topological equations

3.3.1. Void fraction topological equation

We first average the topological equation (2.7) for the PIF. As the fraction of presence of phase k is defined as the average of the PIF (Eq. 3.8), the average of (2.7) gives an evolution equation for α_k. After some simple manipulations, and using Eq. (2.19), the following equation is obtained:

$$\frac{\partial \alpha_k}{\partial t} + \nabla.\left(\alpha_k \overline{\underline{v}_k}^{k}\right) - \left\langle \chi_k \nabla.\underline{v}_k \right\rangle = \left\langle \frac{\dot{m}_k}{\rho_k} \delta_I \right\rangle \tag{3.58}$$

It can be seen that this equation resembles, but is not identical to, the mass balance equation (3.12). We can make use of the local instantaneous mass balance (2.16) rewritten under the following form:

$$\nabla.\underline{v}_k = -\frac{1}{\rho_k} \frac{d_k \rho_k}{dt} \tag{3.59}$$

where we used the following definition of the microscopic material derivative:

$$\frac{d_k}{dt} \equiv \frac{\partial}{\partial t} + \underline{v}_k.\nabla \tag{3.60}$$

based on the local instantaneous velocity $\underline{v}_k$, which is different from the macroscopic material derivative (3.33) based on the averaged velocity $\overline{\overline{\underline{v}_k}}^{k}$. The equation (3.58) becomes:

$$\frac{\partial \alpha_k}{\partial t} + \nabla.\left(\alpha_k \overline{\underline{v}_k}^{k}\right) + \left\langle \frac{\chi_k}{\rho_k} \frac{d_k \rho_k}{dt} \right\rangle = \left\langle \frac{\dot{m}_k}{\rho_k} \delta_I \right\rangle \tag{3.61}$$

This equation can be compared to the mass balance equation (3.12) which can be rewritten:

$$\frac{\partial \alpha_k}{\partial t} + \nabla.\left(\alpha_k \overline{\overline{\underline{v}_k}}^{k}\right) + \frac{\alpha_k}{\overline{\rho_k}^{k}} \frac{D_k \overline{\rho_k}^{k}}{Dt} = \frac{\Gamma_k}{\overline{\rho_k}^{k}} \tag{3.62}$$

It can be seen that the two equations (3.61) and (3.62) are equivalent if, and only if, the density of phase k remains constant, i.e. phase k is incompressible. If the density of phase k varies on the microscopic scale, then $\overline{\overline{\underline{v}_k}}^{k} \neq \overline{\underline{v}_k}^{k}$ and (3.61) and (3.62) are not equivalent. The first one is an equation for the void fraction which is distinct from the mass balance equation, the differences coming from local compressibility variations.

3.3.2. interfacial area topological equation

A second topological equation is given by the local instantaneous interfacial area concentration (IAC) balance equation (2.29). Its average gives the following IAC balance equation:

$$\frac{\partial a_I}{\partial t} + \nabla . \left[a_I \overline{\underline{w}}^I \right] = a_I \overline{\Gamma_s}^I \tag{3.63}$$

where a_I is the average IAC defined by the following relation:

$$a_I \equiv \left\langle \delta_I \right\rangle \tag{3.64}$$

The quantities $\overline{\underline{w}}^I$ and $\overline{\Gamma_s}^I$ are the interfacial averaged interface velocity and source term per unit surface. This last quantity is due to numerous physical phenomena like coalescence and ruptures, phase change, interfacial instabilities and so on… Nevertheless, it is not easy to introduce discontinuous phenomena, like coalescence and break-up, in the general context presented here. They will be introduced later, in a section specific to dispersed two-phase flows, like bubbly or droplet flows.

3.4. Turbulence equations

Despite the fact that it is quite difficult to define “true turbulence” in two-phase flows, because the fluctuations observed in the bulk phases are also strongly coupled to the erratic movements of interfaces, authors often follow the same “line of reasoning” to derive balance equations for turbulence in two-phase flows than in single-phase flows. We are concerned here with RANS approaches (RANS means Reynolds Averaged Navier-Stokes) which have nothing to see with LES (Large Eddy Simulation) dedicated to more fundamental studies. The different RANS approaches can be classified according to the number of balance equations used to calculate the averaged effect of turbulence. Typical approaches use zero, one, two or even more balance equations to describe turbulence, the model having the greater number of equations being probably the Reynolds Stress Model (RSM) which uses seven turbulence equations (e.g. Schiestel, 1993). Here we follow the classical (single-phase) approach to derive balance equations for turbulence (Schiestel, 1993), and simply extend these equations to two-phase flows by considering the functions in the sense of generalized functions, and using the tools developed in section 2.

The more important mean quantity associated to turbulence is probably the Reynolds stress tensor which has been defined in (3.18). Adopting the notations of Schiestel (1993), we prefer to redefine the Reynolds stress tensor as the double velocity correlation:

$$\underline{\underline{R}}_k \equiv \overline{\overline{\underline{v}'_k \underline{v}'_k}}^k \tag{3.65}$$

The original stress tensor (3.18) can be retrieved by simply make $\underline{\underline{\tau}}_k^T \equiv -\overline{\rho_k}^k \underline{\underline{R}}_k$. The Reynolds stress tensor balance equations are obtained by the following method. First of all, we derive the equation for the fluctuating velocity $\underline{v}'_k$ which is defined by Eq. (3.16). This can be done by subtracting the equation for the mean velocity (3.21) from the equation for the local instantaneous velocity (2.25). In order to do this, we make the simplifying assumption of an incompressible phase k, hence here:

$$\text{Simplifying assumption}: \rho_k = \text{cte} = \overline{\rho_k}^k \quad \Rightarrow \quad \overline{\overline{\underline{v}_k}}^k = \overline{\underline{v}_k}^k \tag{3.66}$$

Hence it is not useful, in this paragraph, to make the distinction between the Favre average and the phase average. Under the assumption of a constant density, Eq. (2.25) can be rewritten:

$$\frac{\partial \underline{v}_k}{\partial t} + \nabla.\left(\underline{v}_k \underline{v}_k\right) = -\nabla \frac{p_k}{\rho_k} + \nabla.\frac{\underline{\underline{\tau}}_k}{\rho_k} + \underline{g} \tag{3.67}$$

Under the same assumption, Eq. (3.21) can be rewritten:

$$\begin{aligned} &\frac{\partial \overline{\overline{\underline{v}_k}}^k}{\partial t} + \nabla.\left(\overline{\overline{\underline{v}_k}}^k \, \overline{\overline{\underline{v}_k}}^k\right) = -\frac{\overline{\overline{\underline{v}_k}}^k}{\alpha_k}\frac{D_k \alpha_k}{Dt} - \nabla\left(\frac{\overline{p_k}^k}{\rho_k}\right) - \frac{\overline{p_k}^k}{\alpha_k \rho_k}\nabla\alpha_k \\ &+ \nabla.\left(\frac{\overline{\underline{\underline{\tau}}_k}^k}{\rho_k}\right) - \nabla.\underline{\underline{R}}_k + \frac{\overline{\underline{\underline{\tau}}_k}^k}{\alpha_k \rho_k}.\nabla\alpha_k - \underline{\underline{R}}_k.\frac{\nabla\alpha_k}{\alpha_k} + \underline{g} + \frac{\underline{M}_k}{\alpha_k \rho_k} \end{aligned} \tag{3.68}$$

Subtracting (3.68) from (3.67), the following equation is obtained:

$$\begin{aligned} &\frac{\partial \underline{v}'_k}{\partial t} + \nabla.\left(\overline{\overline{\underline{v}_k}}^k \, \underline{v}'_k + \underline{v}'_k \overline{\overline{\underline{v}_k}}^k + \underline{v}'_k \underline{v}'_k\right) = -\nabla \frac{p'_k}{\rho_k} + \nabla.\frac{\underline{\underline{\tau}}'_k}{\rho_k} + \frac{\overline{\overline{\underline{v}_k}}^k}{\alpha_k}\frac{D_k \alpha_k}{Dt} + \frac{\overline{p_k}^k}{\alpha_k \rho_k}\nabla\alpha_k \\ &+ \nabla.\underline{\underline{R}}_k - \frac{\overline{\underline{\underline{\tau}}_k}^k}{\alpha_k \rho_k}.\nabla\alpha_k + \underline{\underline{R}}_k.\frac{\nabla\alpha_k}{\alpha_k} - \frac{\underline{M}_k}{\alpha_k \rho_k} \end{aligned} \tag{3.69}$$

where we have defined the fluctuating pressure and viscous stress tensor as in the following relations:

$$\begin{aligned} p'_k &\equiv p_k - \overline{p_k}^k \\ \underline{\underline{\tau}}'_k &\equiv \underline{\underline{\tau}}_k - \overline{\underline{\underline{\tau}}_k}^k \end{aligned} \tag{3.70}$$

The second step to obtain the equation for the typical component $R_{k,ij}$ is to make:

$$\overline{\overline{v'_{k,i} \times \text{Eq.}\left(v'_{k,j}\right) + v'_{k,j} \times \text{Eq.}\left(v'_{k,i}\right)}}^k \tag{3.71}$$

This gives:

$$\begin{aligned}
&\overline{\overline{\frac{\partial v'_{k,i}v'_{k,j}}{\partial t}}}^{k}+\overline{\overline{v_{k,l}}}^{k}\,\overline{\overline{\frac{\partial v'_{k,i}v'_{k,j}}{\partial x_l}}}^{k}+2\overline{\overline{v'_{k,i}v'_{k,j}}}^{k}\,\frac{\partial \overline{\overline{v_{k,l}}}^{k}}{\partial x_l}=-\overline{\overline{v_{k,i}}}^{k}\,\overline{\overline{v'_{k,j}\frac{\partial v'_{k,l}}{\partial x_l}}}^{k}-\overline{\overline{v_{k,j}}}^{k}\,\overline{\overline{v'_{k,i}\frac{\partial v'_{k,l}}{\partial x_l}}}^{k}\\
&-\overline{\overline{v'_{k,j}v'_{k,l}}}^{k}\,\frac{\partial \overline{\overline{v_{k,i}}}^{k}}{\partial x_l}-\overline{\overline{v'_{k,i}v'_{k,l}}}^{k}\,\frac{\partial \overline{\overline{v_{k,j}}}^{k}}{\partial x_l}-\overline{\overline{2v'_{k,i}v'_{k,j}\frac{\partial v'_{k,l}}{\partial x_l}}}^{k}-\overline{\overline{v'_{k,j}v'_{k,l}\frac{\partial v'_{k,i}}{\partial x_l}}}^{k}-\overline{\overline{v'_{k,i}v'_{k,l}\frac{\partial v'_{k,j}}{\partial x_l}}}^{k}\\
&-\overline{\overline{v'_{k,j}\frac{\partial}{\partial x_i}\left(\frac{p'_k}{\rho_k}\right)}}^{k}-\overline{\overline{v'_{k,i}\frac{\partial}{\partial x_j}\left(\frac{p'_k}{\rho_k}\right)}}^{k}+\overline{\overline{v'_{k,j}\frac{\partial}{\partial x_l}\left(\frac{\tau'_{k,il}}{\rho_k}\right)}}^{k}+\overline{\overline{v'_{k,i}\frac{\partial}{\partial x_l}\left(\frac{\tau'_{k,jl}}{\rho_k}\right)}}^{k}
\end{aligned}\qquad(3.72)$$

Now we must take care that the phase averaging operator $\overline{\overline{\;\;}}^{k}$ cannot be simply permuted with the space and time derivatives, contrary to the ensemble averaging operator < > (Eq. 3.4). Therefore we must come back to the ensemble averaging operator to permute it with the two derivatives:

$$\begin{aligned}
\overline{\frac{\partial\psi_k}{\partial t}}^{k}&=\frac{\alpha_k}{\alpha_k}\overline{\frac{\partial\psi_k}{\partial t}}^{k}=\frac{1}{\alpha_k}\left\langle\chi_k\frac{\partial\psi_k}{\partial t}\right\rangle=\frac{1}{\alpha_k}\frac{\partial}{\partial t}\left\langle\chi_k\psi_k\right\rangle-\frac{1}{\alpha_k}\left\langle\psi_k\frac{\partial\chi_k}{\partial t}\right\rangle\\
&=\frac{1}{\alpha_k}\frac{\partial}{\partial t}\left(\alpha_k\overline{\psi_k}^{k}\right)-\frac{1}{\alpha_k}\left\langle\psi_k\,\underline{w}.\underline{n}_k\delta_I\right\rangle
\end{aligned}\qquad(3.73)$$

$$\begin{aligned}
\overline{\frac{\partial\psi_k v_{k,l}}{\partial x_l}}^{k}&=\frac{\alpha_k}{\alpha_k}\overline{\frac{\partial\psi_k v_{k,l}}{\partial x_l}}^{k}=\frac{1}{\alpha_k}\left\langle\chi_k\frac{\partial\psi_k v_{k,l}}{\partial x_l}\right\rangle=\frac{1}{\alpha_k}\frac{\partial}{\partial x_l}\left\langle\chi_k\psi_k v_{k,l}\right\rangle-\frac{1}{\alpha_k}\left\langle\psi_k v_{k,l}\frac{\partial\chi_k}{\partial x_l}\right\rangle\\
&=\frac{1}{\alpha_k}\frac{\partial}{\partial x_l}\left(\alpha_k\overline{\psi_k v_{k,l}}^{k}\right)+\frac{1}{\alpha_k}\left\langle\psi_k\,\underline{v}_k.\underline{n}_k\delta_I\right\rangle
\end{aligned}\qquad(3.74)$$

We recall that in the context of the assumption (3.66), we make no distinction between the Favre average $\overline{\overline{\;\;}}^{k}$ and the phase average $\overline{\;\;}^{k}$. In this context, we can work on the different terms of Eq. (3.72). First we note that:

$$2\overline{\overline{v'_{k,i}v'_{k,j}}}^{k}\,\frac{\partial\overline{\overline{v_{k,l}}}^{k}}{\partial x_l}+\overline{\overline{2v'_{k,i}v'_{k,j}\frac{\partial v'_{k,l}}{\partial x_l}}}^{k}=\overline{\overline{2v'_{k,i}v'_{k,j}\frac{\partial v_{k,l}}{\partial x_l}}}^{k}=0\qquad(3.75)$$

because of the incompressibility of phase k.

$$
\begin{aligned}
&\overline{\overline{\frac{\partial v'_{k,i}v'_{k,j}}{\partial t}}}^{k} + \overline{\overline{v_{k,l}}}^{k}\,\overline{\overline{\frac{\partial v'_{k,i}v'_{k,j}}{\partial x_l}}}^{k} + \overline{\overline{v'_{k,j}v'_{k,l}\frac{\partial v'_{k,i}}{\partial x_l}}}^{k} + \overline{\overline{v'_{k,i}v'_{k,l}\frac{\partial v'_{k,j}}{\partial x_l}}}^{k} = \\
&= \overline{\overline{\frac{\partial v'_{k,i}v'_{k,j}}{\partial t}}}^{k} + \overline{\overline{\frac{\partial v'_{k,i}v'_{k,j}v_{k,l}}{\partial x_l}}}^{k} = \\
&= \frac{1}{\alpha_k}\frac{\partial}{\partial t}\left(\alpha_k R_{k,ij}\right) + \frac{1}{\alpha_k}\frac{\partial}{\partial x_l}\left(\alpha_k\,\overline{v'_{k,i}v'_{k,j}v_{k,l}}^{k}\right) - \frac{1}{\alpha_k}\left\langle v'_{k,i}v'_{k,j}\left(\underline{w}-\underline{v}_k\right).\underline{n}_k\delta_I\right\rangle \\
&= \frac{1}{\alpha_k}\frac{\partial}{\partial t}\left(\alpha_k R_{k,ij}\right) + \frac{1}{\alpha_k}\frac{\partial}{\partial x_l}\left(\alpha_k\left(R_{k,ij}\overline{\overline{v_{k,l}}}^{k} + \overline{v'_{k,i}v'_{k,j}v'_{k,l}}^{k}\right)\right) - \frac{1}{\alpha_k}\left\langle v'_{k,i}v'_{k,j}\frac{\dot{m}_k}{\rho_k}\delta_I\right\rangle
\end{aligned}
\tag{3.76}
$$

where the definition (3.65) has been used. Eq. (3.72) can be rewritten:

$$
\begin{aligned}
&\frac{1}{\alpha_k}\frac{\partial}{\partial t}\left(\alpha_k R_{k,ij}\right) + \frac{1}{\alpha_k}\frac{\partial}{\partial x_l}\left(\alpha_k R_{k,ij}\overline{\overline{v_{k,l}}}^{k}\right) = \frac{1}{\alpha_k}\left\langle v'_{k,i}v'_{k,j}\frac{\dot{m}_k}{\rho_k}\delta_I\right\rangle - \frac{1}{\alpha_k}\frac{\partial}{\partial x_l}\left(\alpha_k\,\overline{v'_{k,i}v'_{k,j}v'_{k,l}}^{k}\right) \\
&\quad - R_{k,jl}\frac{\partial \overline{\overline{v_{k,i}}}^{k}}{\partial x_l} - R_{k,il}\frac{\partial \overline{\overline{v_{k,j}}}^{k}}{\partial x_l} - \overline{\overline{v_{k,i}}}^{k}\,\overline{\overline{v'_{k,j}\frac{\partial v'_{k,l}}{\partial x_l}}}^{k} - \overline{\overline{v_{k,j}}}^{k}\,\overline{\overline{v'_{k,i}\frac{\partial v'_{k,l}}{\partial x_l}}}^{k} \\
&\quad - \overline{\overline{v'_{k,j}\frac{\partial}{\partial x_i}\left(\frac{p'_k}{\rho_k}\right)}}^{k} - \overline{\overline{v'_{k,i}\frac{\partial}{\partial x_j}\left(\frac{p'_k}{\rho_k}\right)}}^{k} + \overline{\overline{v'_{k,j}\frac{\partial}{\partial x_l}\left(\frac{\tau'_{k,il}}{\rho_k}\right)}}^{k} + \overline{\overline{v'_{k,i}\frac{\partial}{\partial x_l}\left(\frac{\tau'_{k,jl}}{\rho_k}\right)}}^{k}
\end{aligned}
\tag{3.77}
$$

Defining the interfacial Reynolds stress tensor weighted by phase change by the following relation:

$$
\Gamma_k R^{\Gamma}_{k,ij} \equiv \left\langle v'_{k,i}v'_{k,j}\dot{m}_k\delta_I\right\rangle
\tag{3.78}
$$

Eq. (3.77) can be rewritten:

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\alpha_k R_{k,ij}\right)+\frac{\partial}{\partial x_l}\left(\alpha_k R_{k,ij}\overline{\overline{v_{k,l}}}^{k}\right)=\frac{\Gamma_k}{\rho_k}R^{\Gamma}_{k,ij} && \text{(I)}\\
&\quad-\frac{\partial}{\partial x_l}\left(\alpha_k\,\overline{v'_{k,i}v'_{k,j}v'_{k,l}}^{k}\right) && \text{(II)}\\
&\quad-\alpha_k\left(R_{k,jl}\frac{\partial\overline{\overline{v_{k,i}}}^{k}}{\partial x_l}-R_{k,il}\frac{\partial\overline{\overline{v_{k,j}}}^{k}}{\partial x_l}\right) && \text{(III)}\\
&\quad-\alpha_k\overline{\overline{v_{k,i}}}^{k}\,\overline{\overline{v'_{k,j}\frac{\partial v'_{k,l}}{\partial x_l}}}^{k}-\alpha_k\overline{\overline{v_{k,j}}}^{k}\,\overline{\overline{v'_{k,i}\frac{\partial v'_{k,l}}{\partial x_l}}}^{k} && \text{(IV)}\\
&\quad-\alpha_k\overline{\overline{v'_{k,j}\frac{\partial}{\partial x_i}\left(\frac{p'_k}{\rho_k}\right)}}^{k}-\alpha_k\overline{\overline{v'_{k,i}\frac{\partial}{\partial x_j}\left(\frac{p'_k}{\rho_k}\right)}}^{k} && \text{(V)}\\
&\quad+\alpha_k\overline{\overline{v'_{k,j}\frac{\partial}{\partial x_l}\left(\frac{\tau'_{k,il}}{\rho_k}\right)}}^{k}+\alpha_k\overline{\overline{v'_{k,i}\frac{\partial}{\partial x_l}\left(\frac{\tau'_{k,jl}}{\rho_k}\right)}}^{k} && \text{(VI)}
\end{aligned}
\tag{3.79}
$$

The LHS of Eq. (3.79) is simply the transport term of the correlation $R_{k,ij}$ by the mean velocity field. The different terms in the RHS have been numbered: the physical significance of each of them is given hereafter. (I) is due to phase change (Reynolds stress interfacial transfer associated to mass transfer). (II) is the divergence of the triple correlation which needs to be modelled. (III) are the production terms by the mean velocity gradients, they need no further modelling. (IV) are additional terms due to the “compressibility” of the fluctuating velocity field, these terms do not exist in incompressible single-phase flows, they are specific to two-phase flows. (V) contain the divergence of the pressure-velocity correlations as well as the pressure-deformation correlations. At the end, (VI) contain the molecular diffusion as well as the viscous dissipation terms.

Another often used balance equations in turbulence models is the turbulent kinetic energy equation. Recalling the definition (3.35) of the turbulent kinetic energy, it can be seen that the trace of the Reynolds stress tensor is twice this turbulent kinetic energy:

$$
R_{k,ii}=\mathrm{tr}\left(\underline{\underline{R}}_k\right)=2K_k \tag{3.80}
$$

Therefore, the balance equation for turbulent kinetic energy can be obtained by taking half of the trace of the equation (3.79):

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\alpha_k K_k\right)+\frac{\partial}{\partial x_l}\left(\alpha_k K_k \overline{\overline{v_{k,l}}}^{k}\right)=\frac{\Gamma_k}{\rho_k}K_k^{\Gamma} && \text{(I)}\\
&-\frac{\partial}{\partial x_l}\left(\alpha_k \overline{\frac{v'_{k,i}v'_{k,i}}{2}v'_{k,l}}^{k}\right) && \text{(II)}\\
&-\alpha_k R_{k,il}\frac{\partial \overline{\overline{v_{k,i}}}^{k}}{\partial x_l} && \text{(III)}\\
&-\alpha_k \overline{\overline{v_{k,i}}}^{k}\ \overline{\overline{v'_{k,i}\frac{\partial v'_{k,l}}{\partial x_l}}}^{k} && \text{(IV)}\\
&-\alpha_k \overline{\overline{v'_{k,i}\frac{\partial}{\partial x_i}\left(\frac{p'_k}{\rho_k}\right)}}^{k} && \text{(V)}\\
&+\alpha_k \overline{\overline{v'_{k,i}\frac{\partial}{\partial x_l}\left(\frac{\tau'_{k,il}}{\rho_k}\right)}}^{k} && \text{(VI)}
\end{aligned}
\tag{3.81}
$$

with the same interpretation of the different terms in the RHS of Eq. (3.81). The six equations (3.79) are the basis of the two-phase RSM (Reynolds Stress Model) and the equation (3.81) is the basis of the two-phase K-ε model. The same ε equation must be derived to complete each of these models. Due to its complexity, this balance equation will not be developed here (See e.g. Morel, 1995).

4. Hybrid approach for dispersed two-phase flows

When the flow is dispersed (e.g. a bubbly or a droplet flow), one of the two phases (the discrete or dispersed phase) is embedded into the other phase (the continuous phase) under the form of discrete inclusions. These inclusions can be fluid inclusions (like bubbles in a liquid or droplets in a gas) or solid ones (like in dusty flows for example). If the volumetric fraction of the dispersed phase α_d is very small ($\alpha_d << 1$), the flow is said to be dilute, otherwise it is said to be dense. Due to the dissymmetry of the two phases for this kind of flows, it is advantageous to describe the discrete phase by tools different from those for the continuous phase. Up to now (sections 2-3), the two phases were treated symmetrically, without any assumption on their common interface configuration. Here, we take part of the existence of these numerous inclusions to derive the equations for the dispersed phase from population balances. As these population balances have no sense for the continuous phase, we are obliged to treat the continuous phase as in the preceding section. The two phases are therefore described in different manners, hence the name “hybrid” of the approach.

In comparison to the symmetric two-fluid model developed in section 3, the advantages to proceed like that are numerous:

- The equations for the dispersed phase are more readable, because they resemble to the equations developed for a single particle.

- The existing knowledge on a single particle behaviour inside a fluid can be utilized to help to the closure of the averaged equations (e.g. the force exerted on a particle by the surrounding fluid, see paragraph 2.7.3, its deformation rate…).
- The number of freedom degrees to describe the dispersed phase can be considerably reduced in simple cases. For example, for spherical rigid particles, it is sufficient to introduce the three components of the particle translation velocity, together with the three components of its rotation velocity to describe its motion completely. This reduction of the number of freedom degrees allows to reduce the description complexity when it is possible and to avoid spurious effects typically encountered with the use of the two-fluid model.
- Some important phenomena, like inter-particle collisions or fluid particles coalescence and break-up, are much easier to introduce.

The basic tool is the introduction of a distribution function $f(\underline{\xi};\underline{x},t)$ to describe the fluid (or solid) particles population. The vector $\underline{\xi}$ groups the so-called *phase internal coordinates*, separated from the *external coordinates* $\underline{x}$ and t by a semicolon (;). This vector contains all the variables chosen to describe the population, like the particles positions, their translational velocities, their size and shape, temperatures… We must distinguish between a N-particle description and a 1 particle description, the second one being a particular case from the first. Of course, the complexity of the description must be adapted to the complexity of the problem under study. This is one of the great advantages of the method to allow to start from a simple description and to progressively increase the complexity of this description by adding other effects, conserving the previously acquired knowledge. We start by the description of a population of spherical particles of identical sizes, a simple case studied independently by Laviéville (1997) and Kaufmann (2004).

4.1. Description of a population mono-disperse in size but multi-disperse in velocity

4.1.1. Definition of variables and derivation of the main balance equations

This description is analogous to the kinetic theory of gas molecules. We assume a dispersion of spherical particles having a common constant size, given by their radius a or their diameter d = 2a. To each particle is associated a position and velocity realization in phase coordinates. The averaging operator < > must be understood as an ensemble average conditioned on one continuous phase realization. We assume that the particle diameter is so small that flow around each particle can be considered as a creeping flow, so that the Stokes drag given by Eq. (2.84) for a solid particle or the drag (2.85) for a clean bubble can be used. This last simplification is only for pedagogical purpose, and will be relaxed later.

The particle distribution function $f(\underline{c};\underline{x},t)$ is defined such that $f(\underline{c};\underline{x},t)d^3c d^3x$ is the probable number of particle centres at time t located in a volume element d^3x around point $\underline{x}$ and having a translation velocity between $\underline{c}$ and $\underline{c}+d\underline{c}$. The rotation velocity is not considered for simplicity in this first description. Let us consider a field quantity $\psi_d(\underline{c};\underline{x},t)$ characterizing the dispersed phase. It can be a scalar, a vector or a tensor. The associated averaged field is defined by the following relation:

$$n(\underline{x},t)\langle\psi_d\rangle_n \equiv \int \psi_d(\underline{c};\underline{x},t) f(\underline{c};\underline{x},t) d^3c \tag{4.1}$$

where $\langle\psi_d\rangle_n$ is the n-weighted average of the quantity $\psi_d(\underline{c};\underline{x},t)$, $n(\underline{x},t)$ being the particle number density which is defined by:

$$n(\underline{x},t) \equiv \int f(\underline{c};\underline{x},t) d^3c \tag{4.2}$$

An example of quantity $\psi_d(\underline{c};\underline{x},t)$ is the particle velocity itself, which average gives the mean particle velocity:

$$n(\underline{x},t)\underline{U}_d(\underline{x},t) \equiv n(\underline{x},t)\langle\underline{c}\rangle_n = \int \underline{c} f(\underline{c};\underline{x},t) d^3c \tag{4.3}$$

The product of the particle number density and the mean velocity is then given by the first order moment of the velocity distribution function. Other useful moments are the p-order centred moments:

$$n(\underline{x},t)\langle c'_i \ldots c'_k\rangle_n = \int \left[(c_i - U_{d,i})\ldots(c_k - U_{d,k})\right] f(\underline{c};\underline{x},t) d^3c \tag{4.4}$$

The second order centred velocity moment is called the *kinetic stress tensor* and is quite analogous for the dispersed phase to the Reynolds stress tensor for the continuous phase (see section 3).

It is possible to derive the equation for f in a very general manner (e.g. Achard, 1978). When the particle velocity is the single internal phase coordinate, this equation reads:

$$\frac{\partial f}{\partial t} + \nabla.(f\,\underline{c}) + \frac{\partial\left(f\left\langle \frac{F_j}{m}\middle|\underline{c}\right\rangle\right)}{\partial c_j} = \frac{\partial f}{\partial t}\Big|_{coll} \tag{4.5}$$

Where $\underline{F}$ is the sum of the forces exerted on the particle, including its weight. The ratio $\underline{F}/m$ is therefore the particle acceleration, m being the particle mass, and $\left\langle \frac{\underline{F}}{m}\middle|\underline{c}\right\rangle$ is the conditional average of the particle acceleration, having a particle with velocity $\underline{c}$. The RHS of Eq. (4.5) comes from inter-particle collisions. Eq. (4.5) is called a Liouville-Boltzmann equation. Multiplying (4.5) by the quantity $\psi_d(\underline{c};\underline{x},t)$ and integrating over the velocity space, the following Enskog general equation is deduced:

$$\begin{gathered}\frac{\partial n\langle\psi_d\rangle_n}{\partial t} + \nabla.\left(n\langle\psi_d\underline{c}\rangle_n\right) = n\left\langle\frac{F_j}{m}\frac{\partial\psi_d}{\partial c_j}\right\rangle_n + C(\psi_d) + n\left\langle\frac{\partial\psi_d}{\partial t} + \underline{c}.\nabla\psi_d\right\rangle_n \quad \text{with :}\\ C(\psi_d) \equiv \int \psi_d \frac{\partial f}{\partial t}\Big|_{coll} d^3c\end{gathered} \tag{4.6}$$

where $C(\psi_d)$ results from collisions. In what follows, we derive the equations for the zero-th, first and second order moments of the velocity distribution function. These three moments correspond to the particle number density balance equation, the particle momentum balance equation and the kinetic stress tensor balance equation.

- Particle number density balance equation: $\psi_d = 1$

Making $\psi_d = 1$ in Eq. (4.6), the following particle number density balance equation is obtained:

$$\frac{\partial n}{\partial t} + \nabla.\left(n\underline{U}_d\right) = C(1) \tag{4.7}$$

Excluding particle break-up and coalescence, as well as nucleation and collapse, the RHS of (4.7) is zero. If there is no phase change, the particle mass m is a constant. Multiplying (4.7) by m and remarking that $nm = \alpha_d\rho_d$, the particle mass balance equation is retrieved:

$$\frac{\partial \alpha_d\rho_d}{\partial t} + \nabla.\left(\alpha_d\rho_d\underline{U}_d\right) = 0 \tag{4.8}$$

which is the same than (3.12) when k = d and assuming no phase change ($\Gamma_k = 0$).

- Particle momentum balance equation: $\psi_d = \underline{c}$

Making $\psi_d = \underline{c}$ in Eq. (4.6), the following particle momentum balance equation is obtained:

$$\frac{\partial n\underline{U}_d}{\partial t} + \nabla.\left(n\langle\underline{c}\underline{c}\rangle_n\right) = n\left\langle\frac{\underline{F}}{m}\right\rangle_n + C(\underline{c}) \tag{4.9}$$

Multiplying by the constant particle mass m and taking into account that $nm = \alpha_d\rho_d$, Eq. (4.9) can be rewritten:

$$\frac{\partial \alpha_d\rho_d\underline{U}_d}{\partial t} + \nabla.\left(\alpha_d\rho_d\underline{U}_d\underline{U}_d\right) = -\nabla.\left(\alpha_d\rho_d\langle\underline{c}'\underline{c}'\rangle_n\right) + \alpha_d\rho_d\left\langle\frac{\underline{F}}{m}\right\rangle_n + C(m\underline{c}) \tag{4.10}$$

where we have introduced the kinetic stress tensor $\langle\underline{c}'\underline{c}'\rangle_n$ which is a particular form of Eq. (4.4).

- Kinetic stress tensor balance equation: $\psi_d = (c_i - U_{d,i}(\underline{x},t))\,(c_j - U_{d,j}(\underline{x},t))$

Making $\psi_d = (c_i - U_{d,i}(\underline{x},t))\,(c_j - U_{d,j}(\underline{x},t))$ in Eq. (4.6), the following kinetic stress tensor balance equation is obtained:

$$\begin{aligned}\frac{\partial n\left\langle c'_i c'_j\right\rangle_n}{\partial t}+\frac{\partial\left(n\left\langle c'_i c'_j\right\rangle_n U_{d,k}\right)}{\partial x_k}=-\frac{\partial\left(n\left\langle c'_i c'_j c'_k\right\rangle_n\right)}{\partial x_k}+n\left\langle\frac{F_k}{m}\left(\delta_{ik}c'_j+\delta_{jk}c'_i\right)\right\rangle_n+C(c'_i c'_j)\\-n\left\langle c'_i c'_k\right\rangle_n\frac{\partial U_{d,j}}{\partial x_k}-n\left\langle c'_j c'_k\right\rangle_n\frac{\partial U_{d,i}}{\partial x_k}\end{aligned}\tag{4.11}$$

or, multiplying by the constant particle mass m:

$$\begin{aligned}\frac{\partial\alpha_d\rho_d\left\langle c'_i c'_j\right\rangle_n}{\partial t}+\frac{\partial\left(\alpha_d\rho_d\left\langle c'_i c'_j\right\rangle_n U_{d,k}\right)}{\partial x_k}=-\frac{\partial\left(\alpha_d\rho_d\left\langle c'_i c'_j c'_k\right\rangle_n\right)}{\partial x_k}\\-\alpha_d\rho_d\left\langle c'_i c'_k\right\rangle_n\frac{\partial U_{d,j}}{\partial x_k}-\alpha_d\rho_d\left\langle c'_j c'_k\right\rangle_n\frac{\partial U_{d,i}}{\partial x_k}+\alpha_d\rho_d\left\langle\frac{F_i}{m}c'_j+\frac{F_j}{m}c'_i\right\rangle_n+C(mc'_i c'_j)\end{aligned}\tag{4.12}$$

4.1.2. Closure of the interfacial force for creeping flows

When the flow at the particle scale (microscopic scale) can be considered as a creeping flow, and the assumptions of the section 2.7 are globally satisfied, we can use the results of the section 2.7.3 to express the force $\underline{F}$ exerted by the continuous phase on the particle. Assuming a solid particle which is heavier than the surrounding fluid, the main forces exerted on the particle are its weight and the Stokes drag given by Eqs. (2.83)-(2.84). Manipulating these expressions, the following expression is found for the particle acceleration:

$$\frac{\underline{F}}{m}=\underline{g}+\frac{\underline{u}-\underline{c}}{\tau_p}\quad\text{with}\quad\tau_p\equiv\frac{\rho_d d^2}{18\mu_c}\tag{4.13}$$

where τ_p is the so-called *particle relaxation time*. The vector $\underline{u}$ is the continuous phase velocity at the particle location (the so-called fluid velocity *seen* by the particle). It should be kept in mind that the result (4.13) is valid only when the particle is a solid one and the flow surrounding the particle can be described by the creeping flow theory, hence the particle Reynolds number defined by (2.82) must be much smaller than 1. Introducing the result (4.13) into the momentum equation (4.10) gives:

$$\frac{\partial\alpha_d\rho_d\underline{U}_d}{\partial t}+\nabla.\left(\alpha_d\rho_d\underline{U}_d\underline{U}_d\right)=-\nabla.\left(\alpha_d\rho_d\left\langle\underline{c'c'}\right\rangle_n\right)+\alpha_d\rho_d\underline{g}+\alpha_d\rho_d\frac{\left\langle\underline{u}\right\rangle_n-\underline{U}_d}{\tau_p}+C(m\underline{c})\tag{4.14}$$

where we have assumed a constant particle relaxation time. The averaged velocity $\left\langle\underline{u}\right\rangle_n$ is the n-weighted average of the fluid velocity seen by the particles. Its modelling is quite delicate, and this quantity is often decomposed by introducing the following dispersion (or drift) velocity (e.g. Laviéville, 1997):

$$\left\langle\underline{u}\right\rangle_n=\underline{U}_c+\underline{V}_d\tag{4.15}$$

where $\underline{V}_d$ is the macroscopic dispersion velocity and $\underline{U}_c$ is the continuous mean velocity (in the usual sense of the two-fluid model). Hence, Eq. (4.14) can be rewritten:

$$\frac{\partial \alpha_d \rho_d \underline{U}_d}{\partial t} + \nabla.\left(\alpha_d \rho_d \underline{U}_d \underline{U}_d\right) = -\nabla.\left(\alpha_d \rho_d \left\langle \underline{c'c'} \right\rangle_n\right) + \alpha_d \rho_d \underline{g} - \alpha_d \rho_d \frac{\underline{U}_d - \underline{U}_c - \underline{V}_d}{\tau_p} + C(m\underline{c}) \tag{4.16}$$

According to Jackson (1997), the collision term appearing in the momentum balance for the particles can be rewritten by using the following Taylor development:

$$C(m\underline{c}) = \nabla.\left(nm\left\langle \underline{\underline{s}} \right\rangle_n\right) - \frac{1}{2}\nabla\nabla : \left(nm\left\langle \underline{\underline{\underline{s}}} \right\rangle_n\right) + \ldots \tag{4.17}$$

where $\left\langle \underline{\underline{s}} \right\rangle_n$ and $\left\langle \underline{\underline{\underline{s}}} \right\rangle_n$ are averaged collisional stress tensors of order 2 and 3 respectively. The equation (4.17) is a Taylor development. For it to be valid, the successive terms in this development should become smaller and smaller. Therefore, assuming that the second term in Eq. (4.17) is negligibly small in comparison to the first, and substituting the remaining term in Eq. (4.16) gives:

$$\frac{\partial \alpha_d \rho_d \underline{U}_d}{\partial t} + \nabla.\left(\alpha_d \rho_d \underline{U}_d \underline{U}_d\right) = -\nabla.\left(\alpha_d \rho_d \left(\left\langle \underline{\underline{s}} \right\rangle_n + \left\langle \underline{c'c'} \right\rangle_n\right)\right) + \alpha_d \rho_d \underline{g} - \alpha_d \rho_d \frac{\underline{U}_d - \underline{U}_c - \underline{V}_d}{\tau_p} \tag{4.18}$$

We can compare Eq. (4.18) to the classical two-fluid model momentum equation (3.21) written for the dispersed phase (k = d). Assuming that the mean particle velocity $\underline{U}_d$ can be assimilated to the phase averaged velocity $\overline{\overline{\underline{v}_d}}^{d}$ (which is just approximately true because $\overline{\overline{\underline{v}_d}}^{d}$ takes into account all the particle internal motions as well as the particle rotation, which are not taken into account in $\underline{U}_d$. This illustrates the simplifications brought by the kinetic theory approach developed here in comparison to the two-fluid approach), we can compare the RHS of Eqs. (3.21) and (4.18). The average pressure and viscous stresses for the continuous phase are replaced by the collisional stresses for the dispersed phase. In the same manner, the Reynolds stress tensor for the continuous phase is replaced by the kinetic stress tensor for the dispersed phase. At the end, we see that an approximated form of the momentum interfacial transfer (valid under the assumptions retained in this paragraph) is given by:

$$\underline{M}_d \cong -\alpha_d \rho_d \frac{\underline{U}_d - \underline{U}_c - \underline{V}_d}{\tau_p} \tag{4.19}$$

The continuous phase interfacial momentum transfer term $\underline{M}_c$ is then deduced from the interfacial momentum balance (3.30) and an appropriate expression for the mixture momentum source $\underline{M}_m$.

The kinetic stress tensor obeys to the balance equation (4.12). Using Eq. (4.13), the term involving the interfacial force in (4.12) can be rewritten as:

$$\begin{aligned}&\alpha_d\rho_d\left\langle \frac{F_i}{m}c'_j+\frac{F_j}{m}c'_i\right\rangle_n = \alpha_d\rho_d\left\langle \frac{u_i-c_i}{\tau_p}c'_j+\frac{u_j-c_j}{\tau_p}c'_i\right\rangle_n = \\ &=\frac{\alpha_d\rho_d}{\tau_p}\left\langle u'_ic'_j-c'_ic'_j+u'_jc'_i-c'_jc'_i\right\rangle_n = 2\frac{\alpha_d\rho_d}{\tau_p}\left[\left\langle\frac{u'_ic'_j+u'_jc'_i}{2}\right\rangle_n-\left\langle c'_ic'_j\right\rangle_n\right]\end{aligned}\tag{4.20}$$

The relation (4.20) involves the kinetic stress tensor $\left\langle c'_ic'_j\right\rangle_n$ as well as the symmetric part of the fluid-particle velocity correlation tensor $\left\langle u'_ic'_j\right\rangle_n$. Substituting (4.20) into (4.12) gives:

$$\begin{aligned}&\frac{\partial\alpha_d\rho_d\left\langle c'_ic'_j\right\rangle_n}{\partial t}+\frac{\partial\left(\alpha_d\rho_d\left\langle c'_ic'_j\right\rangle_n U_{d,k}\right)}{\partial x_k}=-\frac{\partial\left(\alpha_d\rho_d\left\langle c'_ic'_jc'_k\right\rangle_n\right)}{\partial x_k} && \text{(I)}\\ &-\alpha_d\rho_d\left\langle c'_ic'_k\right\rangle_n\frac{\partial U_{d,j}}{\partial x_k}-\alpha_d\rho_d\left\langle c'_jc'_k\right\rangle_n\frac{\partial U_{d,i}}{\partial x_k} && \text{(II)}\\ &+2\frac{\alpha_d\rho_d}{\tau_p}\left[\left\langle\frac{u'_ic'_j+u'_jc'_i}{2}\right\rangle_n-\left\langle c'_ic'_j\right\rangle_n\right]_n && \text{(III)}\\ &+C(mc'_ic'_j) \qquad \text{(IV)}\end{aligned}\tag{4.21}$$

The LHS of Eq. (4.21) is simply the transport of the kinetic stress tensor component $\left\langle c'_ic'_j\right\rangle_n$ by the mean dispersed phase velocity. The four terms in the RHS of Eq. (4.21) have the following physical significance: (I): divergence of the triple velocity correlation. The balance equation for the triple velocity correlation $\left\langle c'_ic'_jc'_k\right\rangle_n$ could be derived from the Enskog general equation (4.6). This term is very similar to the term (II) in the RHS of the Reynolds stress tensor balance equation (3.79) developed in the context of the two-fluid model. The two terms (II) in Eq. (4.21) are production terms by the mean velocity gradients. They need no further modelling. They are also very similar to the production terms (III) in the RHS of Eq. (3.79). The term (III) in the RHS of Eq. (4.21) represents the interaction with the turbulent motion of the continuous phase. The last term (IV) is due to inter-particle collisions.

4.2. Description of a population multi-disperse in size (and velocity)

In this section, a second internal phase coordinate, the particle diameter d, is introduced. The fluid or solid particles are assumed to retain the spherical shape, but their diameter varies from one particle to another one, and can also vary along one particle's trajectory. The particle distribution function $f(d,\underline{c};\underline{x},t)$ is now defined such that $f(d,\underline{c};\underline{x},t)\delta d\delta^3c\delta^3x$ is the probable number of particle centres at time t located in a volume element δ^3x around point $\underline{x}$, having a diameter between d and d + δd and having a translation velocity between $\underline{c}$ and $\underline{c}+\delta\underline{c}$. The Liouville-Bolztmann equation (4.5) is generalized into the following equation:

$$\frac{\partial f}{\partial t}+\nabla.\left(f\,\underline{c}\right)=-\frac{\partial}{\partial c_j}\left(f\left\langle\frac{F_j}{m}\middle|\underline{c},d\right\rangle\right)-\frac{\partial}{\partial d}\left(f\left\langle\frac{Dd}{Dt}\middle|\underline{c},d\right\rangle\right)+\left.\frac{df}{dt}\right|_{coll}\tag{4.22}$$

The n-weighted average quantity (4.1) is simply generalized into the following definition:

$$n(\underline{x},t)\langle\psi_d\rangle_n(\underline{x},t)\equiv\int\psi_d(d,\underline{c};\underline{x},t)f(d,\underline{c};\underline{x},t)\underbrace{\delta d\delta^3 c}_{\delta\Omega} \tag{4.23}$$

where $\delta\Omega$ is an abbreviated notation for the internal phase space element $\delta d\delta^3 c$. When particles have different sizes, their masses are also different, and one has advantage to introduce the following Favre (or mass weighted) average for the dispersed phase:

$$\alpha(\underline{x},t)\rho(\underline{x},t)\langle\psi_d\rangle_\alpha(\underline{x},t)\equiv\int m(d;\underline{x},t)\psi_d(d,\underline{c};\underline{x},t)f(d,\underline{c};\underline{x},t)\delta\Omega \tag{4.24}$$

where α and ρ are related to the dispersed phase without ambiguity on the notation, since the dispersed phase is the only one considered here. The particle mass m is given by:

$$m(d;\underline{x},t)\equiv\rho(\underline{x},t)\frac{\pi d^3}{6} \tag{4.25}$$

due to the spherical particles assumption.

Multiplying Eq. (4.22) by $m\psi$ and integrating the resulting equation over the internal phase space, the following equation is obtained:

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\alpha\rho\langle\psi\rangle_\alpha\right)-\int\frac{\partial m\psi}{\partial t}f\delta\Omega+\nabla.\left(\alpha\rho\langle\psi\underline{c}\rangle_\alpha\right)-\int\underline{c}.\nabla(m\psi)f\delta\Omega=\\&=+\int\frac{\partial m\psi}{\partial c_j}\left\langle\frac{Dc_j}{Dt}\middle|d,\underline{c}\right\rangle f\delta\Omega+\int\frac{\partial m\psi}{\partial d}\left\langle\frac{Dd}{Dt}\middle|d,\underline{c}\right\rangle f\delta\Omega+C(m\psi)\end{aligned} \tag{4.26}$$

Where $\left\langle\frac{Dc_j}{Dt}\middle|d,\underline{c}\right\rangle$ is the jth component of the particle acceleration, it is given by:

$$\left\langle\frac{Dc_j}{Dt}\middle|d,\underline{c}\right\rangle=\left\langle\frac{F_j}{m}\middle|\underline{c},d\right\rangle \tag{4.27}$$

For geometrical moments (quantities not weighted by the particle mass), an equation analogous to Eq. (4.26) can be obtained by multiplying Eq. (4.22) by ψ (instead of $m\psi$) then integrating. This gives:

$$\begin{aligned}&\frac{\partial}{\partial t}\left(n\langle\psi\rangle_n\right)-\int\frac{\partial\psi}{\partial t}f\delta\Omega+\nabla.\left(n\langle\psi\underline{c}\rangle_n\right)-\int\underline{c}.\nabla\psi f\delta\Omega=\\&=+\int\frac{\partial\psi}{\partial c_j}\left\langle\frac{Dc_j}{Dt}\middle|d,\underline{c}\right\rangle f\delta\Omega+\int\frac{\partial\psi}{\partial d}\left\langle\frac{Dd}{Dt}\middle|d,\underline{c}\right\rangle f\delta\Omega+C(\psi)\end{aligned} \tag{4.28}$$

4.2.1. Mass and momentum balance equations

In order to derive the mass and momentum balance equations for the dispersed phase, one must first derive expressions for the Lagrangian derivatives of the particle diameter Dd/Dt and velocity $D\underline{c}/Dt$ appearing in Eqs. (4.26) and (4.28). To fix the ideas, we will assume that the considered particles are gas bubbles in a continuous liquid. The first derivative represents the diameter variation measured along the bubble path. This size variation is due to the gas compressibility on one hand, and on the phase change (vaporization or condensation) on the other hand. The bubble mass variation Dm/Dt is only due to phase change, hence we can write:

$$\frac{Dm}{Dt} = \frac{D}{Dt}\left(\rho\frac{\pi d^3}{6}\right) = \dot{m}\pi d^2 \tag{4.29}$$

where $\dot{m}$ is defined as the bubble mass gain per unit surface per unit time due to phase change. $\dot{m}$ is the mean value of (2.19) over the bubble surface. The Lagrangian derivative D/Dt being taken at the bubble velocity $\underline{c}$, it is easy to deduce from (4.29):

$$\frac{Dd}{Dt} = -\frac{d}{3\rho}\left(\frac{\partial\rho}{\partial t} + \underline{c}.\nabla\rho\right) + 2\frac{\dot{m}}{\rho} \tag{4.30}$$

For the sake of generality, we will also extend the interaction force model for bubbles in liquid with any value of the bubble Reynolds number. The study of the forces exerted on a bubble by the surrounding liquid is a quite difficult task. According to Minier & Peirano (2001), a general form for the bubble momentum equation can be proposed which retains drag, pressure gradient, added mass and gravity forces (see also Maxey & Riley, 1983; Gatignol, 1983). Another important force called the lift force (Auton, 1987) is missing in the description of Minier & Peirano (2001), but for our pedagogical purpose, it is sufficient to start from the work of Minier & Peirano (2001). The momentum balance equation of a single bubble reads:

$$\rho\frac{\pi d^3}{6}\frac{D\underline{c}}{Dt} = \underbrace{\rho\frac{\pi d^3}{6}\underline{g}}_{\substack{\text{bubble}\\ \text{weigth}}} \underbrace{-\rho_c\frac{\pi d^3}{6}\underline{g} + \rho_c\frac{\pi d^3}{6}\frac{D\underline{u}}{Dt}}_{\text{generalized Archimede's force}} \underbrace{-\frac{1}{2}\rho_c C_D\frac{\pi d^2}{4}\left|\underline{c}-\underline{u}\right|\left(\underline{c}-\underline{u}\right)}_{\text{drag force}} \underbrace{-C_A\rho_c\frac{\pi d^3}{6}\left[\frac{D\underline{c}}{Dt}-\frac{D\underline{u}}{Dt}\right]}_{\text{added}-\text{mass force}} \tag{4.31}$$

Four forces are taken into account in the RHS of Eq. (4.31), these are the bubble weight, the generalized Archimede’s force, the drag force and the added mass force. The vector field $\underline{u}$ is the liquid velocity seen by the bubble (the so-called unperturbed velocity). The drag force is a generalized expression of the force already seen for creeping flows (section 2.7.3) and involves a drag coefficient C_D which needs to be modelled as a function of the bubble diameter, bubble Reynolds number… Empirical expressions for C_D can be found in Ishii (1990). The last force, the added mass force, is proportional to the liquid mass displaced by the bubble immersion (ρ_c denoting the continuous phase density, i.e. the liquid density). This force is a brake to the bubble acceleration relatively to the surrounding liquid. The added mass coefficient C_A essentially depends on the bubble shape. For spherical bubbles, it is equal to one half ($C_A = ½$). Dividing Eq. (4.31) by the bubble mass and regrouping the terms proportional to the bubble acceleration in the LHS gives the following equation:

$$\left(1+C_A\frac{\rho_c}{\rho}\right)\frac{Dc_i}{Dt}=\frac{u_i-c_i}{\tau_p}+\left(1+C_A\right)\frac{\rho_c}{\rho}\frac{Du_i}{Dt}+\left(1-\frac{\rho_c}{\rho}\right)g_i \tag{4.32}$$

where τ_p is a bubble relaxation time analogous to the one defined in Eq. (4.13) for Stokes drag, and is given here as a function of the drag coefficient:

$$\tau_p \equiv \frac{4}{3}\frac{\rho}{\rho_c}\frac{d}{C_D\left|c-u\right|} \tag{4.33}$$

Introducing the definitions of the following coefficients:

$$\tau_p^m \mathrel{\hat{=}} \tau_p\left(1+C_A\frac{\rho_c}{\rho}\right), \quad b \mathrel{\hat{=}} \frac{\left(1+C_A\right)\frac{\rho_c}{\rho}}{\left(1+C_A\frac{\rho_c}{\rho}\right)}, \quad c \mathrel{\hat{=}} \frac{\left(1-\frac{\rho_c}{\rho}\right)}{\left(1+C_A\frac{\rho_c}{\rho}\right)} \tag{4.34}$$

The following final form of the bubble acceleration is obtained:

$$\frac{Dc_i}{Dt}=\frac{u_i-c_i}{\tau_p^m}+b\frac{Du_i}{Dt}+cg_i \tag{4.35}$$

Now we can derive the averaged mass and momentum balance equations for the bubble swarm. Making $\psi = 1$ in Eq. (4.26) gives:

$$\frac{\partial}{\partial t}(\alpha\rho)-\int\frac{\partial m}{\partial t}f\delta\Omega+\nabla.\left(\alpha\rho\left\langle\underline{c}\right\rangle_\alpha\right)-\int\underline{c}.\nabla(m)f\delta\Omega=\int\frac{\partial m}{\partial d}\frac{Dd}{Dt}f\delta\Omega \tag{4.36}$$

where we put $C(m) = 0$ because the bubble mass is a collisional invariant. Introducing (4.30) into (4.36) and simplifying yields:

$$\frac{\partial\alpha\rho}{\partial t}+\nabla.\left(\alpha\rho\left\langle\underline{c}\right\rangle_\alpha\right)=\int\pi d^2\dot{m}f\delta\Omega\equiv\Gamma_d \tag{4.37}$$

which is analogous to the mass balance equation (3.12) obtained in the context of the two-fluid model.

Making $\psi = \underline{c}$ in Eq. (4.26) gives:

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\alpha\rho\left\langle\underline{c}\right\rangle_\alpha\right)-\int\frac{\partial m\underline{c}}{\partial t}f\delta\Omega+\nabla.\left(\alpha\rho\left\langle\underline{c}\underline{c}\right\rangle_\alpha\right)-\int\underline{c}.\nabla\left(m\underline{c}\right)f\delta\Omega=\\&\quad=+\int\frac{\partial m\underline{c}}{\partial c_j}\frac{Dc_j}{Dt}f\delta\Omega+\int\frac{\partial m\underline{c}}{\partial d}\frac{Dd}{Dt}f\delta\Omega+C\left(m\underline{c}\right)\end{aligned} \tag{4.38}$$

First substituting (4.30) into (4.38) and simplifying gives:

$$\frac{\partial}{\partial t}\left(\alpha\rho\langle\underline{c}\rangle_{\alpha}\right)+\nabla.\left(\alpha\rho\langle\underline{\underline{c c}}\rangle_{\alpha}\right)=\int m\frac{D\underline{c}}{Dt}f\delta\Omega+\int\dot{m}\pi d^{2}\underline{c}f\delta\Omega+C\left(m\underline{c}\right) \tag{4.39}$$

The first term in the RHS is the bubble acceleration term, the bubble acceleration being given by Eq. (4.35). The second term in the RHS is the averaged recoil force and the last term is the collision term which can be rewritten as the divergence of a tensor (see Eq. 4.17). Defining an average dispersed velocity weighted by phase change:

$$\int\dot{m}\pi d^{2}\underline{c}f\delta\Omega\equiv\Gamma_{d}\underline{U}_{d}^{\Gamma} \tag{4.40}$$

and developing the average of the tensorial product of velocities in the LHS of (4.39) as the product of the averaged velocities plus the average of the product of fluctuating velocities, as it has been done in sections 3.2.2 and 4.1.1, Eq. (4.39) becomes:

$$\begin{gathered}\frac{\partial}{\partial t}\left(\alpha\rho\underline{U}_{\alpha}\right)+\nabla.\left(\alpha\rho\underline{U}_{\alpha}\underline{U}_{\alpha}\right)=-\nabla.\left(\alpha\rho\left(\langle\underline{\underline{c'c'}}\rangle_{\alpha}+\langle\underline{\underline{s}}\rangle_{n}\right)\right)+\Gamma_{d}\underline{U}_{d}^{\Gamma}+\alpha\rho\left\langle\frac{D\underline{c}}{Dt}\right\rangle_{\alpha}\\ \text{with}:\underline{U}_{\alpha}\equiv\langle\underline{c}\rangle_{\alpha}\end{gathered} \tag{4.41}$$

Introducing Eq. (4.35) into the last term of (4.41), we find:

$$\frac{\partial}{\partial t}\left(\alpha\rho\underline{U}_{\alpha}\right)+\nabla.\left(\alpha\rho\underline{U}_{\alpha}\underline{U}_{\alpha}\right)=-\nabla.\left(\alpha\rho\left(\langle\underline{\underline{c'c'}}\rangle_{\alpha}+\langle\underline{\underline{s}}\rangle_{n}\right)\right)+\Gamma_{d}\underline{U}_{d}^{\Gamma}+\alpha\rho\left[\left\langle\frac{\underline{u}-\underline{c}}{\tau_{p}^{m}}\right\rangle_{\alpha}+b\left\langle\frac{D\underline{u}}{Dt}\right\rangle_{\alpha}+c\underline{g}\right] \tag{4.42}$$

The momentum balance (4.42) is the generalized form of (4.18) is the case of multi-size bubbles and including phase change and the added-mass effect. Its use needs further modelling.

4.2.2. Geometrical moments balance equations

Defining the pth order moment of the diameter distribution function by the following relation:

$$S_{p}\equiv\int d^{p}f\delta\Omega=n\langle d^{p}\rangle_{n} \tag{4.43}$$

The balance equation for the moment S_p is obtained by making $\psi = d^p$ in the general equation (4.28). The result is:

$$\frac{\partial S_{p}}{\partial t}+\nabla.\left(n\langle d^{p}\underline{c}\rangle_{n}\right)=-\frac{p}{3\rho}\left(S_{p}\frac{\partial\rho}{\partial t}+n\langle d^{p}\underline{c}\rangle_{n}.\nabla\rho\right)+\frac{2p}{\rho}\int d^{p-1}\dot{m}f\delta\Omega+C\left(d^{p}\right) \tag{4.44}$$

Several useful equations can be derived from (4.44) by taking different values for p. It is clear from Eq. (4.43) that the zeroth order moment S_0 corresponds to the bubble number density n. The void fraction α is proportional to the third order moment ($\alpha = \pi S_3/6$) as it can be seen by making $\psi_d = 1$ in Eq. (4.24). In the present context of the kinetic theory, the interfacial area concentration a_I (Eq. 3.64) is proportional to the second order moment ($a_I = \pi S_2$). We can also define an infinite number of mean bubble diameters by the following relation:

$$d_{pq} \equiv \left(\frac{S_p}{S_q} \right)^{\frac{1}{p-q}} \tag{4.45}$$

The mathematical form for the bubble diameter distribution function can then be presumed to close the modelling of the bubble diameters. For example, Kamp et al. (2001) choose to model the marginal distribution function $f(d;\underline{x},t)$ by a log-normal law:

$$f(d;\underline{x},t) = n(\underline{x},t) \frac{1}{\sqrt{2\pi}\hat{\sigma} d} \exp\left(-\frac{\left[\log(d/d_{00})\right]^2}{2\hat{\sigma}^2} \right) \tag{4.46}$$

where d_{00} is the median diameter and $\hat{\sigma}$ is a width parameter. These two parameters depend on the position $\underline{x}$ and time t and, together with n, completely define the bubble diameter distribution function at these position and time. Their (exact) closure needs to know the void fraction α together with the two particular moments S_1 and S_2 through the relations:

$$\hat{\sigma} = \sqrt{\ln\left(\frac{6\alpha S_1}{\pi S_2{}^2} \right)}, \qquad d_{00} = \frac{6\alpha}{\pi S_2} e^{-5\hat{\sigma}^2/2} \tag{4.47}$$

Hence one needs to calculate S_1 and S_2 by using their balance equations:

$$\begin{aligned} &\frac{\partial S_1}{\partial t} + \nabla.\left(n\left\langle d\underline{c} \right\rangle_n \right) = -\frac{1}{3\rho}\left(S_1 \frac{\partial \rho}{\partial t} + n\left\langle d\underline{c} \right\rangle_n .\nabla\rho \right) + \frac{2}{\rho}\int \dot{m} f \delta\Omega + C(d) \\ &\frac{\partial S_2}{\partial t} + \underbrace{\nabla.\left(n\left\langle d^2\underline{c} \right\rangle_n \right)}_{\text{I}} = \underbrace{-\frac{2}{3\rho}\left(S_2 \frac{\partial \rho}{\partial t} + n\left\langle d^2\underline{c} \right\rangle_n .\nabla\rho \right)}_{\text{II}} + \underbrace{\frac{4}{\rho}\int d\dot{m} f \delta\Omega}_{\text{III}} + \underbrace{C(d^2)}_{\text{IV}} \end{aligned} \tag{4.48}$$

The equations (4.48) can be solved numerically when all the terms I, II, III and IV have been modelled by appropriate closure laws. These terms correspond to (I) transport of the moment by the mean and fluctuating velocities, (II) gas compressibility, (III) phase change and (IV) coalescence, breakup, nucleation and collapse. The modelling of all these effects is a complex task that will not be developed in this introduction to two-phase flows.

Of course, the expression (4.46) for the bubble diameter distribution function is only one candidate. Other expressions, more or less complex, have been proposed in the literature to model the spectrum of the bubble diameters.

5. Various special cases

5.1. Terminal velocity

Due to the buoyancy (Archimede) force, the relative bubble motion in a liquid is essentially upwards (bubble rising velocity). The terminal velocity is defined as the particular bubble rising velocity at equilibrium between the drag force and the Archimede force, i.e. when all the other forces have cessed to act on the bubble. This is a very important notion since it allows obtaining the bubble drag coefficient by measuring the terminal velocity which is accessible by visual observation. The link between the drag coefficient and the terminal velocity is simply obtained by equalizing the drag force and the net buoyancy force (the Archimede force together with the bubble weight). For a single (isolated) bubble in a liquid, this gives (by simplifying Eq. 4.31):

$$(\rho_L - \rho_G)\frac{\pi d^3}{6} g = \frac{1}{2}\rho_L C_{D\infty}\frac{\pi d^2}{4} v_{T\infty}{}^2 \tag{5.1}$$

where the index ∞ recalls that the bubble is alone in a infinite liquid medium. The terminal velocity can be calculated directly from (5.1) if the drag coefficient is known:

$$v_{T\infty} = \sqrt{\frac{4(\rho_L - \rho_G)gd}{3\rho_L C_{D\infty}}} \tag{5.2}$$

For a very small bubble, the creeping flow assumption can be done (section 2.7.3), and the drag coefficient is given by Eq. (2.85). Substitution of (2.85) into (5.2) gives the value of the terminal velocity of a clean bubble in creeping flow:

$$v_{T\infty} = \frac{(\rho_L - \rho_G)gd^2}{12\mu_L} \tag{5.3}$$

The relations (5.1)-(5.3) have been obtained for a single bubble case. Now we will search an equivalent notion for a bubbly flow with a large number of bubbles starting from the two fluid model momentum equations (3.32). We assume that the two phases do not accelerate and that the molecular and Reynolds stress tensors can be neglected. We also assume no phase change. Our last assumption is to assume that all the different average pressures appearing in Eq. (3.32) are equal to a single average pressure p, an assumption commonly used in two-fluid codes. Under these assumptions, the equations (3.32) written for the two phases degenerate into the following ones:

$$\begin{aligned} 0 &= -\alpha_G \nabla p + \alpha_G \overline{\rho_G}^G \underline{g} + \underline{M}_{iG} \\ 0 &= -\alpha_L \nabla p + \alpha_L \overline{\rho_L}^L \underline{g} + \underline{M}_{iL} \end{aligned} \tag{5.4}$$

We can eliminate the mean pressure gradient between the two equations (5.4) by multiplying the first one by α_L, the second one by α_G and then subtracting. This yield:

$$\alpha_G \alpha_L \left(\overline{\rho_L}^L - \overline{\rho_G}^G\right)\underline{g} = \alpha_L \underline{M}_{iG} - \alpha_G \underline{M}_{iL} \tag{5.5}$$

Neglecting the differences between $\underline{M}_{ik}$ and $\underline{M}_k$ (which is coherent with the assumptions previously made) and neglecting the momentum interfacial source $\underline{M}_m$ (i.e. neglecting surface tension), we can write $\underline{M}_{iG} = - \underline{M}_{iL} = \underline{M}_G$ hence:

$$\alpha_G \alpha_L \left(\overline{\rho_L}^L - \overline{\rho_G}^G \right) \underline{g} = \underline{M}_G \tag{5.6}$$

The LHS of Eq. (5.6) contains the buoyancy forces. The RHS contains the interfacial forces (drag, added mass…). Considering that the drag force is the only one to act on each bubble, because we are searching for the average terminal velocity, the RHS of (5.6) must be replaced by the average drag force per unit volume of the two-phase mixture. This averaged drag force is given by the equation (4.19) together with the equation (4.33). Neglecting the dispersion velocity $\underline{V}_d$ and replacing $\underline{U}_d - \underline{U}_c = \underline{U}_G - \underline{U}_L$ by the mean terminal velocity V_T, the equation (5.6) becomes:

$$\alpha_G \alpha_L \left(\overline{\rho_L}^L - \overline{\rho_G}^G \right) \underline{g} = \frac{3}{4} \alpha_G \overline{\rho_L}^L \frac{C_D}{d} V_T^{\,2} \tag{5.7}$$

where all the bubbles are assumed to have the same diameter d. The mean terminal velocity can be obtained directly from (5.7):

$$V_T = \sqrt{\frac{4\alpha_L \left(\overline{\rho_L}^L - \overline{\rho_G}^G \right) g d}{3 \overline{\rho_L}^L C_D}} \tag{5.8}$$

The comparison of the two expressions (5.2) and (5.8) of the terminal velocity shows the appearance of a factor $\alpha_L^{1/2}$ in the mean terminal velocity, which was not appear in the terminal velocity of a single bubble. First it should be remarked that in the case of an isolated bubble in a infinite liquid medium, the liquid fraction of presence α_L tends to 1 hence the difference between (5.2) and (5.8) in the single bubble case is negligible. For a bubble swarm, it should be remarked that the Archimede force exerted on a given bubble comes from the bubbly mixture surrounding the considered bubble and not from the liquid alone. Due to the presence of the other bubbles, the bubbly mixture is lighter than the liquid as it is characterized by the following density:

$$\rho_M \equiv \alpha_G \rho_G + \alpha_L \rho_L \tag{5.9}$$

which is the averaged form of the first equation (2.11). Replacing ρ_L in the LHS of (5.1) by the mixture density (5.9), the result (5.8) is retrieved in place of (5.2), showing the complete compatibility between the two fluid model and a balance of forces on a given bubble, at least in the simple conditions studied here.

6. The example of the NEPTUNE_CFD code

In this section, we give the example of the NEPTUNE_CFD code developed by EDF and CEA for the numerical studies of two-phase flows in simple as well as in complex geometries. This code uses a generalized two-fluid model, generalized meaning that it can consider more than two phases. However, for the sake of simplicity, we will consider only the case of two phases here. Nowadays, this code has essentially three types of applications: gas-solid flows (solid particles dispersed in a continuous gas phase), and two types of gas-liquid flows: stratified flows and boiling bubbly flows. In order to fix the ideas, we will consider only the last application (boiling bubbly flows). The basic set of equations solved to simulate such a flow is a six-fold one. It contains:

- Two mass balance equations
- Two momentum balance equations
- Two energy balance equations in the form of total enthalpy balance equations

This basic set of equations is compulsory to calculate boiling bubbly flows. To these six equations, auxiliary balance equations can be added. These additional balance equations are mainly of two types:

- Turbulence equations used to predict the turbulence existing in each phase
- Geometrical moments balance equations used to predict the average topology of the bubble swarm.

In what follows, we describe each of these equations in their “simplified” version used in the code, together with the main assumptions in order to obtain these simplified equations from the exact balance equations developed in the preceding sections.

6.1. The mass balance equations in the NEPTUNE_CFD code

The mass balance equations are given by the equation (3.12) together with the jump condition (3.13). The RHS of this equation is due to phase change (vaporization or condensation). The only additional precision we must give is that this RHS is split into two different contributions. The first contribution is the phase change through the surfaces of the already existing bubbles. The second contribution is the nucleation of new bubbles. Nucleation can classically be divided into wall nucleation (heterogeneous nucleation) and nucleation in the liquid bulk (homogeneous nucleation). Here, only the heterogeneous nucleation is considered and modelled. Therefore, the equation (3.12) is rewritten in the following form:

$$\frac{\partial \alpha_k \overline{\rho_k}^k}{\partial t} + \nabla.\left(\alpha_k \overline{\rho_k}^k \overline{\overline{\underline{v}_k}}^k \right) = \Gamma_k^c + \Gamma_k^{nuc} = \Gamma_k \tag{6.1}$$

where Γ_k^c is the part of Γ_k associated to phase change other than nucleation and Γ_k^{nuc} is the nucleation part. The part Γ_k^c can represent vaporization or condensation but the nucleation part Γ_k^{nuc} only corresponds to vaporization. Obviously, each part of Γ_k verifies the relation (3.13) independently.

6.2. The momentum balance equations in the NEPTUNE_CFD code

The momentum balance equations in the NEPTUNE_CFD code are based on the equation (3.32) simplified by some additional assumptions. These assumptions are the following ones:

(H1): No distinction is made between the different averaged pressures. All the averaged pressures appearing in Eq. (3.32) are assumed to be identical to a unique mean pressure p.

(H2): No distinction is made between the two averaged velocities $\underline{v}_k^{\Gamma}$ and $\overline{\overline{\underline{v}_k}}^{k}$.

(H3): The last term of Eq. (3.32) is neglected.

Under the assumptions (H1)-(H3), Eq. (3.32) reduces to:

$$\alpha_k \overline{\rho_k}^{k} \frac{D_k \overline{\overline{\underline{v}_k}}^{k}}{Dt} = -\alpha_k \nabla p + \nabla.\left(\alpha_k \left(\overline{\underline{\underline{\tau}}_k}^{k} + \underline{\underline{\tau}}_k^{T}\right)\right) + \alpha_k \overline{\rho_k}^{k} \underline{g} + \underline{M}_{ik} + \alpha_k \overline{\rho_k}^{k} \underline{S}_k \qquad (6.2)$$

The last term of Eq. (6.2) has been added to take into account special physics, like for example the centrifugal force when the flow domain is not Galilean. Now we must give the closure relations for the averaged viscous stress tensor, the Reynolds stress tensor and the interfacial momentum transfer. Assuming that each phase is a Newtonian fluid, the averaged viscous stress tensor can be obtained by properly averaging the closure relation for the local instantaneous (microscopic) viscous stress tensor. This has been done by Ishii (1975) and the result is:

$$\underline{\underline{\tau}}_k = \mu_k \left(\underline{\underline{\nabla \underline{v}_k}} + \underline{\underline{\nabla}}^{T} \underline{v}_k\right) - \frac{2}{3}\mu_k \nabla.\underline{v}_k \underline{\underline{I}}$$

$$\Downarrow$$

$$\alpha_k \overline{\underline{\underline{\tau}}_k}^{k} = \mu_k \left(\underline{\underline{\nabla}}\left(\alpha_k \overline{\underline{v}_k}^{k}\right) + \underline{\underline{\nabla}}^{T}\left(\alpha_k \overline{\underline{v}_k}^{k}\right) - \overline{\overline{\underline{v}_k}}^{k} \nabla\alpha_k - \nabla\alpha_k \overline{\overline{\underline{v}_k}}^{k} + \left\langle \underline{v}'_k \underline{n}_k \delta_I \right\rangle + \left\langle \underline{n}_k \underline{v}'_k \delta_I \right\rangle \right) \qquad (6.3)$$
$$- \frac{2}{3}\mu_k \left(\nabla.\left(\alpha_k \overline{\underline{v}_k}^{k}\right) - \overline{\overline{\underline{v}_k}}^{k} .\nabla\alpha_k + \left\langle \underline{v}'_k .\underline{n}_k \delta_I \right\rangle \right) \underline{\underline{I}}$$

where the following assumption has been made:

(H4): the dynamic viscosity μ_k is a constant.

If we neglect also the fluctuations of the density ρ_k, then there is no distinction between the Favre average velocity $\overline{\overline{\underline{v}_k}}^{k}$ and the phase average velocity $\overline{\underline{v}_k}^{k}$. Therefore, under the following additional assumption:

(H5): the density of phase k does not fluctuate (i.e. it is equal to its average value)

The expression for the averaged viscous stress tensor (6.3) can be simplified:

$$\overline{\underline{\underline{\tau}}_k}^{k} = \mu_k \left(\underline{\underline{\nabla}} \overline{\overline{\underline{v}_k}}^{k} + \underline{\underline{\nabla}}^{T} \overline{\overline{\underline{v}_k}}^{k} + \frac{\left\langle \underline{v}'_k \underline{n}_k \delta_I \right\rangle + \left\langle \underline{n}_k \underline{v}'_k \delta_I \right\rangle}{\alpha_k} \right) - \frac{2}{3}\mu_k \left(\nabla.\overline{\overline{\underline{v}_k}}^{k} + \frac{\left\langle \underline{v}'_k .\underline{n}_k \delta_I \right\rangle}{\alpha_k} \right) \underline{\underline{I}} \qquad (6.4)$$

The terms involving the gradient of the mean velocity are called “bulk deformation tensor” by Ishii (1975) and the additional terms involving the fluctuating velocities together with the interfaces normal and function of presence are called “interfacial extra deformation tensor” by this author. In the NEPTUNE_CFD code, we make the following additional assumption:

(H6): The interfacial extra deformation tensor is neglected and the averaged viscous stress tensor reduces to:

$$\overline{\underline{\underline{\tau}}_k}^k = \mu_k \left(\underline{\underline{\nabla \underline{v}_k}} \overline{\overline{\vphantom{v}}}^k + \underline{\underline{\nabla}}^T \overline{\overline{\underline{v}_k}}^k \right) - \frac{2}{3} \mu_k \nabla . \overline{\overline{\underline{v}_k}}^k \, \underline{\underline{I}} \tag{6.5}$$

Concerning the Reynolds stress tensor $\underline{\underline{\tau}}_k^T$, essentially two modelling approaches are available in the NEPTUNE_CFD code for a continuous phase like the liquid in a bubbly flow. The Reynolds stress tensor can be calculated by solving 6 additional scalar equations for its independent components, like the equations (3.79). We recall that the Reynolds stress tensor is linked to the double velocity correlation tensor (3.65) by the following definition relation:

$$\underline{\underline{\tau}}_k^T \equiv -\overline{\rho_k}^k \, \underline{\underline{R}}_k \tag{6.6}$$

The other method is to make the classical Boussinesq assumption of a turbulent eddy viscosity. Using this assumption, the Reynolds stress tensor is modelled by analogy to the viscous stress tensor by the following relation:

$$\underline{\underline{\tau}}_k^T = \mu_k^T \left(\underline{\underline{\nabla \underline{v}_k}} \overline{\overline{\vphantom{v}}}^k + \underline{\underline{\nabla}}^T \overline{\overline{\underline{v}_k}}^k \right) - \frac{2}{3} \mu_k^T \nabla . \overline{\overline{\underline{v}_k}}^k \, \underline{\underline{I}} - \frac{2}{3} \overline{\rho_k}^k K_k \underline{\underline{I}} \tag{6.7}$$

where μ_k^T is the turbulent eddy viscosity and K_k is the turbulent kinetic energy defined in Eq. (3.35). The closure of these last two quantities will be described in section 6.4.

Other models are available for the turbulence of the dispersed phase, characterized by the kinetic stress tensor introduced in section 4. The most sophisticated models are those based on the kinetic stress tensor balance equation (4.21) but some simpler models, like the algebraic Tchen model (Oesterlé, 2006), can also be used. These models are essentially devoted to dusty flows (gas-solid flows) and not to the boiling bubbly flows described here, therefore we won’t describe these models in more details.

Now we must describe closure laws for the interfacial momentum transfer term $\underline{M}_{ik}$. Due to our assumptions (H1) and (H3), there is no reason to make a distinction between $\underline{M}_{ik}$ and $\underline{M}_k$ (see Eqs. 3.25 and 3.27). Now we will make the following additional assumption:

(H7): the surface tension is not considered

Under this assumption, Eq. (3.30) shows that the gas side interfacial momentum transfer term $\underline{M}_G$ is the opposite of the liquid side interfacial momentum transfer term $\underline{M}_L$. It has been shown in section 4.1.1 that this term can be obtained by dividing the sum of the interfacial forces exerted on a bubble by the bubble mass and then averaging and multiplying by the product $\alpha_d\rho_d$ (Eq. 4.10):

$$\underline{M}_d = \alpha_d \rho_d \left\langle \frac{\underline{F}}{m} \right\rangle_n \tag{6.8}$$

where the index d denotes the dispersed phase. The quantity $\underline{F}$ is the sum of the different forces exerted on a bubble by the surrounding liquid. Classical expressions for these forces can be found in Eq. (4.31). Additional forces like the lift force (Auton, 1987; Tomiyama, 1998) or a wall force (Antal et al., 1991) can also be added. The averaging of all these forces starting from (6.8) is a quite difficult task because all these forces have complicated non linear expressions. Here we will not enter into the details of all the approximations made during this averaging operation. Instead, we simply present the simpler model we use to simulate bubbly flows. In this simple model, the interfacial momentum transfer is the sum of five forces per unit volume of mixture:

$$\underline{M}_{iG} = \underline{M}_G^D + \underline{M}_G^A + \underline{M}_G^L + \underline{M}_G^{TD} + \underline{M}_G^W \tag{6.9}$$

where the five terms in the RHS of Eq. (6.9) denote the averaged drag, added mass, lift, turbulent dispersion and wall forces respectively. The drag force is modelled according to:

$$\underline{M}_G^D = -\underline{M}_L^D = -\frac{1}{8} a_I \rho_L C_D \left| \overline{\overline{\underline{v}_G}}^G - \overline{\overline{\underline{v}_L}}^L \right| \left(\overline{\overline{\underline{v}_G}}^G - \overline{\overline{\underline{v}_L}}^L \right) \tag{6.10}$$

In this equation, a_I is the interfacial area concentration (IAC) which has been defined by Eq. (3.64) in the context of the classical two-fluid model and which is approximately equal to the product πS_2, S_2 being the second order moment of the bubble diameter distribution function (see section 4.2.2). This is a fundamental quantity since it represents the interfacial area available for all the exchanges between phases (mass, momentum and energy). Its modelling for the case of bubbly flows is the object of section 6.5. The drag coefficient C_D is essentially empirical and an abundant literature exists on its modelling. The added mass force is given by the following closure relation:

$$\underline{M}_G^A = -\underline{M}_L^A = -C_A E(\alpha_G) \alpha_G \rho_L \left[\left(\frac{\partial \overline{\overline{\underline{v}_G}}^G}{\partial t} + \overline{\overline{\underline{v}_G}}^G . \underline{\underline{\nabla}} \overline{\overline{\underline{v}_G}}^G \right) - \left(\frac{\partial \overline{\overline{\underline{v}_L}}^L}{\partial t} + \overline{\overline{\underline{v}_L}}^L . \underline{\underline{\nabla}} \overline{\overline{\underline{v}_L}}^L \right) \right] \tag{6.11}$$

where C_A is the added mass coefficient (equal to ½ in the case of spherical bubbles) and the factor $E(\alpha_G)$ has been proposed by Zuber (1964) to take into account the influence of the presence of the other bubbles. It is modelled according to:

$$E(\alpha) = \frac{1 + 2\alpha}{1 - \alpha} \tag{6.12}$$

The factor (6.12) is always greater than 1, therefore the average added mass force on the mean bubble in a bubble swarm is more important than the same force in the case where the bubble is alone in the flow, according to Zuber's proposition.

The lift force has been calculated theoretically by Auton (1987) on a spherical particle immersed in a weakly rotational flow. The averaged expression of this force used in the NEPTUNE_CFD code is:

$$\underline{M}_G^L = -\underline{M}_L^L = -C_L \alpha_G \rho_L \left(\overline{\overline{\underline{v}_G}}^G - \overline{\overline{\underline{v}_L}}^L \right) \wedge \left(\underline{\nabla} \wedge \overline{\overline{\underline{v}_L}}^L \right) \tag{6.13}$$

where C_L is the lift coefficient which is equal to ½ in the theoretical calculation from Auton (1987). In real flows, the lift coefficient varies with the bubble Reynolds number, bubble shape… and can even changes its sign. Tomiyama (1998) gives a non dimensional experimental correlation for the lift coefficient as a function of the bubble Reynolds and Eotvos numbers.

The fourth force is the turbulent dispersion force which has a statistical origin. By statistical, we mean that this force has no equivalent on a single bubble in a quiescent liquid, but comes from the averaging operation which is necessary to pass from the microscopic forces on a given bubble to the macroscopic forces per unit volume of two-phase mixture (Eq. 6.8). All the microscopic forces have complicated non linear expressions and their average introduces some additional terms like the dispersion velocity $\underline{V}_d$ which has been enlighten by averaging the drag force (paragraph 4.1.2). In our simpler model, all these averaged effects coming from non linearities are assumed to be grouped in a single additional force which is called turbulent dispersion force. Several models have been proposed in the literature for this bubble dispersion effect, the simplest having the following mathematical expression:

$$\underline{M}_G^{TD} = -\underline{M}_L^{TD} = -C_{TD} \rho_L K_L \underline{\nabla} \alpha_G \tag{6.14}$$

where C_{TD} is an adjustable coefficient. A modelling of the coefficient C_{TD} is given by Krepper et al. (2006).

6.3. The energy balance equations in the NEPTUNE_CFD code

The NEPTUNE_CFD code uses simplified versions of the thermal energy equation (3.57) and of the total energy jump condition (3.40). Assuming that the heat transfer and phase changes dominate the energy exchanges, the terms arisen from the mechanical effects can be neglected. Under this condition, Eq. (3.57) can be reduced to (Ishii & Hibiki, 2006):

$$\alpha_k \overline{\rho_k}^k \frac{D_k H_k}{Dt} = -\nabla . \left[\alpha_k \left(\overline{\underline{q}_k}^k + \underline{q}_k^T \right) \right] + \alpha_k \overline{Q_k}^k + \Gamma_k \left(H_k^\Gamma - H_k \right) + q''_{kI} a_I \tag{6.15}$$

Ishii & Hibiki (2006) also propose to simplify the total energy jump condition (3.40) by making the following assumptions. The interfacial energy source E_s can be neglected with respect to the large energy exchanges that involve the latent heat at the changing of phases, therefore Eq. (3.40) simplifies to:

$$\sum_k E_k \cong 0 \tag{6.16}$$

The expression (3.53) for E_k needs to be completed by giving constitutive equations for the difference between the mean velocity and the interfacial average velocity weighted by phase change, the turbulent kinetic energies and the interfacial turbulent work. As in two-phase flows with phase changes, the orders of magnitude of these terms compared to the thermal terms is relatively small, these authors assume:

$$\begin{aligned} &H_k - \overline{\overline{h_k}}^{k} \equiv K_k \cong 0 \\ &H_k^\Gamma \cong h_k^\Gamma \\ &\underline{v}_k^\Gamma - \overline{\overline{\underline{v}_k}}^{k} \cong 0 \\ &W_{ki}^T \cong 0 \\ &\overline{p_k}^{I} \cong \overline{p_k}^{k} \end{aligned} \tag{6.17}$$

Then Eq. (3.53) reduces to:

$$E_k = \Gamma_k\left(h_k^\Gamma + \frac{\overline{\overline{v_k}}^{k\,2}}{2}\right) + q''_{kI}a_I - \overline{p_k}^{k}\frac{\partial\alpha_k}{\partial t} + \left(\underline{M}_{ik} - \overline{\underline{\underline{\tau}}_k}^{I}.\nabla\alpha_k\right)\overline{\overline{\underline{v}_k}}^{k} \tag{6.18}$$

And the thermal energy transfer (3.52) can be approximated by:

$$\Lambda_k = \Gamma_k h_k^\Gamma + q''_{kI}a_I - \overline{p_k}^{k}\frac{D_k\alpha_k}{Dt} \tag{6.19}$$

Then, Eq. (6.16) becomes:

$$\sum_k\left(\Gamma_k h_k^\Gamma + q''_{kI}a_I\right) + \Gamma_1\left(\frac{\overline{\overline{v_1}}^{1\,2}}{2} - \frac{\overline{\overline{v_2}}^{2\,2}}{2}\right) - \left(\overline{p_1}^{1} - \overline{p_2}^{2}\right)\frac{\partial\alpha_1}{\partial t} + \sum_k\left(\underline{M}_{ik} - \overline{\underline{\underline{\tau}}_k}^{I}.\nabla\alpha_k\right)\overline{\overline{\underline{v}_k}}^{k} = 0 \tag{6.20}$$

For relatively low speed flow, all the mechanical terms in Eq. (6.20) can be neglected and this equation becomes:

$$\sum_k\left(\Gamma_k h_k^\Gamma + q''_{kI}a_I\right) \cong 0 \tag{6.21}$$

In the NEPTUNE_CFD code, the interfacial heat flux per unit volume per unit time $q''_{kI}a_I$ is denoted by Π'_k. The mass production rate by phase change is obtained by the relation (6.21):

$$\Gamma_2 = -\frac{\Pi'_1 + \Pi'_2}{h_2^\Gamma - h_1^\Gamma} \tag{6.22}$$

The average molecular flux $\overline{\underline{q}_k}^{k}$ can be obtained by averaging the following expression valid for a fluid obeying Fourier's law:

$$\begin{gathered}\underline{q}_k = -\lambda_k \nabla T_k \\ \Downarrow \\ \overline{\underline{q}_k}^k = -\lambda_k \left[\nabla \overline{T_k}^k + \frac{\left\langle \left(T_k - \overline{T_k}^k \right) \underline{n}_k \delta_I \right\rangle}{\alpha_k} \right] = -\lambda_k \left[\nabla \overline{T_k}^k + \frac{\left\langle T'_k \underline{n}_k \delta_I \right\rangle}{\alpha_k} \right]\end{gathered} \tag{6.23}$$

where we made the following assumption:

(H11): the thermal conductivity λ_k is constant or does not fluctuate.

The average molecular flux $\overline{\underline{q}_k}^k$ contains two terms: one proportional to the gradient of the bulk averaged temperature, and the other involving fluctuating temperature at interfaces. The NEPTUNE_CFD code only retains the first of the two terms in (6.23). However the modelling of the second term could be easy (Ishii & Hibiki, 2006). Assuming that the microscopic temperature T_k at interfaces is given by the saturation temperature $T_{sat}(p)$ and that this saturation temperature does not fluctuate (this is the case if it is calculated directly in function of the mean pressure), then the second equation (6.23) can be rewritten approximately:

$$\overline{\underline{q}_k}^k \cong -\lambda_k \left[\nabla \overline{T_k}^k - \left(T_{sat}(p) - \overline{T_k}^k \right) \frac{\nabla \alpha_k}{\alpha_k} \right] \tag{6.24}$$

(H12): the second term in the RHS of (6.24) is not retained.

The turbulent heat flux $\underline{q}_k^T$ is modelled by analogy to Fourier's law for the molecular conduction:

$$\underline{q}_k^T = -\lambda_k^T \nabla \overline{T_k}^k = -\frac{C_{pk} \mu_k^T}{Pr_k^T} \nabla \overline{T_k}^k \tag{6.25}$$

where λ_k^T is a turbulent conductivity linked to the turbulent eddy viscosity μ_k^T by the introduction of a turbulent Prandtl number Pr_k^T.

At the end, the interfacial heat transfers between each phase and the interfaces are modelled according to:

$$\Pi'_k = a_I q''_{kI} \quad \text{with} \quad q''_{kI} = \frac{\lambda_k Nu_k}{d} \left(T_{sat}(p) - \overline{T_k}^k \right) \tag{6.26}$$

where Nu_k is a Nusselt number which must be modelled according to the phase considered and to the situation encountered (overheated or subcooled liquid…). The quantity d in the denominator of (6.26) is a mean bubble diameter. The modelling of bubble diameters and interfacial area concentration a_I are postponed to section 6.5.

6.4. The turbulence balance equations in the NEPTUNE_CFD code

Two major approaches have been developed in the NEPTUNE_CFD code to model the turbulence of a continuous phase like the liquid phase of a bubbly flow. The earlier approach which has been developed is the K-ε approach which is based on a balance equation for the turbulent kinetic energy (Eq. 3.81) and a second balance equation for the turbulent dissipation rate ε_k. A more recent approach is the RSM approach which solves directly the balance equations for the six Reynolds stress components (Eq. 3.79), thus does not need to do a modelling assumption on the Reynolds stress tensor closure, like the Boussinesq hypothesis (6.7). The equations (3.79) are also completed by an additional equation for the turbulent dissipation rate. Due to its extreme complexity, this last equation has not been presented here. The interested reader can find the derivation of the exact equation for ε_k in Morel (1995).

Here we only present the closure of the K-ε model as it is used in the NEPTUNE_CFD code for bubbly flow studies. The Reynolds stress tensor components are determined by the closure relation (6.7) together with the following closure relation for the turbulent eddy viscosity:

$$\mu_k^T = \overline{\rho_k}^k \nu_k^T \quad \text{with} \quad \nu_k^T = C_\mu \frac{{K_k}^2}{\varepsilon_k} \tag{6.27}$$

Two additional balance equations are then written for the turbulent kinetic energy K_k and its dissipation rate ε_k. The turbulent kinetic energy balance equation has a slightly different form from Eq. (3.81). After some manipulations, Eq. (3.81) can be rewritten:

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\alpha_k K_k\right) + \frac{\partial}{\partial x_l}\left(\alpha_k K_k \overline{\overline{v_{k,l}}}^{k}\right) = {} & \underbrace{\frac{\Gamma_k}{\rho_k} K_k^\Gamma}_{\text{I}} \\
& - \frac{\partial}{\partial x_l}\left(\underbrace{\overline{\alpha_k \frac{v'_{k,i} v'_{k,i}}{2} v'_{k,l}}^{k}}_{\text{II}} \underbrace{- \frac{1}{\rho_k} \alpha_k \overline{\overline{v'_{k,i} \tau'_{k,il}}}^{k}}_{\text{III}} \underbrace{+ \frac{\alpha_k}{\rho_k} \overline{\overline{p'_k v'_{k,l}}}^{k}}_{\text{IV}}\right) \\
& \underbrace{- \alpha_k R_{k,il} \frac{\partial \overline{\overline{v_{k,i}}}^{k}}{\partial x_l}}_{\text{V}} \underbrace{- \frac{1}{\rho_k} \left\langle p'_k v'_{k,i} n_{k,i} \delta_I \right\rangle}_{\text{VI}} \underbrace{- \frac{\alpha_k}{\rho_k} \overline{\overline{\tau'_{k,il} \frac{\partial v'_{k,i}}{\partial x_l}}}^{k}}_{\text{VII}} \underbrace{+ \frac{1}{\rho_k} \left\langle \tau'_{k,il} v'_{k,i} n_{k,l} \delta_I \right\rangle}_{\text{VIII}}
\end{aligned} \tag{6.28}$$

with the following interpretation of the eight terms in the RHS of Eq. (6.28):

(I) is the interfacial transfer of turbulent kinetic energy associated to the mass transfer by phase change, its modelling needs to give an expression for K_k^Γ. (II) is the triple velocity correlation, (III) is the phase viscous stress – velocity correlation, (IV) is the phase pressure velocity correlation. (V) is the production term by the mean velocity gradient which needs no further modelling. (VI) and (VIII) are the interfacial pressure-velocity and viscous stress – velocity correlations. The last term (VII) is the turbulent dissipation rate by viscous effect:

$$\tilde{\varepsilon}_k \equiv \frac{1}{\rho_k} \overline{\overline{\tau'_{k,il} \frac{\partial v'_{k,i}}{\partial x_l}}}^{k} \tag{6.29}$$

In fact we do not write an equation for the turbulent dissipation rate (6.29) but an equation for the pseudo turbulent dissipation rate (Schiestel, 1993) which is approximately equal. The pseudo dissipation rate is defined by:

$$\varepsilon_k \equiv \nu_k \overline{\overline{\frac{\partial v'_{k,i}}{\partial x_l} \frac{\partial v'_{k,i}}{\partial x_l}}}^{k} \tag{6.30}$$

In what follows, we give a first proposition of closures of the different unknown terms appearing in Eq. (6.28) which can be used in bubbly flow studies (Morel, 1997). The sum of the three terms (II), (III) and (IV) intervenes in a divergence, therefore making transport of the turbulent kinetic energy without altering it. We choose to model the three terms (II), (III) and (IV) as a whole by introducing a diffusive flux proportional to the gradient of the turbulent kinetic energy:

$$(\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}) = -\frac{\alpha_k}{\rho_k} \frac{\mu_k^T}{\sigma_K} \frac{\partial K_k}{\partial x_l} \tag{6.31}$$

The two terms (VI) and (VIII) are interfacial terms corresponding to the turbulent work of the fluctuating pressure and viscous stresses. We model these two terms collectively by the power developed by the averaged drag in the mean relative velocity:

$$(\mathrm{VI}) + (\mathrm{VIII}) = -\underline{M}_G^D . \left(\overline{\overline{\underline{v}_G}}^{G} - \overline{\overline{\underline{v}_L}}^{L} \right) \equiv P_K^I \tag{6.32}$$

This power corresponds to the additional turbulent kinetic energy produced in the wakes of the bubbles. This wake energy comes from the gravitational potential energy lost by the bubbles during their rise in the surrounding heavier liquid. At the end, we make the following assumption:

(H14): the interfacial average turbulent kinetic energy weighted by phase change K_k^Γ is not different from the mean turbulent kinetic energy K_k:

$$K_k^\Gamma \cong K_k \tag{6.33}$$

Considering the models (6.31)-(6.33), the modelled equation corresponding to (6.28) reads:

$$\begin{aligned} \frac{\partial}{\partial t}(\alpha_k \rho_k K_k) + \nabla . \left(\alpha_k \rho_k K_k \overline{\overline{\underline{v}_k}}^{k} \right) &= \Gamma_k K_k - \alpha_k \rho_k \varepsilon_k - \alpha_k \rho_k \underline{\underline{R}}_k : \overline{\overline{\underline{\underline{\nabla}} \underline{v}_k}}^{k} \\ &+ \nabla . \left(\alpha_k \frac{\mu_k^T}{\sigma_K} \nabla K_k \right) + P_K^I \end{aligned} \tag{6.34}$$

Or, in non conservative form:

$$\alpha_k\rho_k\frac{D_k K_k}{Dt} = -\alpha_k\rho_k\varepsilon_k - \alpha_k\rho_k\underline{\underline{R}}_k : \overline{\overline{\underline{\underline{\nabla}}\underline{v}_k}}^{k} + \nabla.\left(\alpha_k\frac{\mu_k^T}{\sigma_K}\nabla K_k\right) + P_K^I \tag{6.35}$$

where P_K^I is the additional production due to the second phase and modelled by (6.32). It should be noted that in bubbly flows, the equation (6.35) is only used in the liquid phase (k = L). The turbulent dissipation rate balance equation resembles to the equation (6.35). It reads:

$$\alpha_k\rho_k\frac{D_k\varepsilon_k}{Dt} = -C_{\varepsilon 2}\alpha_k\rho_k\frac{{\varepsilon_k}^2}{K_k} - C_{\varepsilon 1}\alpha_k\rho_k\frac{\varepsilon_k}{K_k}\underline{\underline{R}}_k : \overline{\overline{\underline{\underline{\nabla}}\underline{v}_k}}^{k} + \nabla.\left(\alpha_k\frac{\mu_k^T}{\sigma_\varepsilon}\nabla\varepsilon_k\right) + P_\varepsilon^I \tag{6.36}$$

The last term P_ε^I being the interfacial production term of ε_k which is closed by the following relation:

$$P_\varepsilon^I = C_{\varepsilon 3}\frac{\varepsilon_k}{K_k}P_K^I \tag{6.37}$$

6.5. The geometrical moments balance equations in the NEPTUNE_CFD code

The geometrical balance equations used by the NEPTUNE_CFD code are particular and simplified versions of the exact equation (4.44). These geometrical balance equations allow to close the bubble diameter distribution function, hence restoring an average information on the bubbles mean topology. Several approaches have been tempted, according to different presumed mathematical expressions of the bubble diameter distribution function. One of them is the log-normal law which has been presented in section 4.2.2 (Eqs. 4.46 and 4.47). One simpler approach that has been used by many authors is the single size approach where the bubble diameter distribution function is assumed to be given by a Dirac distribution peaked at the Sauter mean diameter d_{32} (all the bubbles are assumed to have the same diameter d_{32} which varies in time and space). By definition, this Sauter mean diameter is given by:

$$d_{32} \equiv \frac{6\alpha_G}{a_I} \tag{6.38}$$

which is a particular case of the general definition (4.45). Hence, in the simplest approach based on d_{32}, it is sufficient to write an additional balance equation for the interfacial area concentration (IAC), the void fraction α_G being part of the solution of the two-fluid model. The IAC is also involved in the modelling of the drag force (Eq. 6.10) as well as in the modelling of the interfacial heat transfers (Eq. 6.26). Hence, due to its particular importance, we will develop here the equation for the IAC. As the interfacial area of a single bubble having a diameter d is given by πd^2, the IAC is given by the following relation:

$$a_I \equiv \int_0^\infty \pi d^2 f(d;\underline{x},t)\partial d \equiv \pi S_2 \tag{6.39}$$

Hence it is equivalent to write an equation for S_2 or for a_I, the last one being obtained by multiplying the first by a factor π. Multiplying Eq. $(4.48)_2$ by π gives:

$$\frac{\partial a_I}{\partial t}+\nabla.\left(n\left\langle \pi d^2\underline{c}\right\rangle_n\right)=-\frac{2}{3\rho}\left(a_I\frac{\partial\rho}{\partial t}+n\left\langle \pi d^2\underline{c}\right\rangle_n.\nabla\rho\right)+\frac{4\pi}{\rho}\int d\dot{m}f\delta d+C\left(\pi d^2\right) \tag{6.40}$$

In this equation $\underline{c}$ denotes the velocity of a bubble having a diameter d. We can decompose this velocity into the mean gas velocity and a fluctuating velocity:

$$\underline{c}=\overline{\overline{\underline{v}_G}}^{G}+\underline{c}'' \tag{6.41}$$

where $\underline{c}''$ is the deviation between the particular bubble velocity and the mean velocity of the bubbles swarm. Introducing Eq. (6.41) into Eq. (6.40) yields:

$$\frac{\partial a_I}{\partial t}+\nabla.\left(a_I\overline{\overline{\underline{v}_G}}^{G}\right)=\underbrace{-\nabla.\left(n\left\langle \pi d^2\underline{c}''\right\rangle_n\right)}_{\text{neglected}}-\frac{2}{3\rho}\left(a_I\frac{D_G\rho}{Dt}+\underbrace{n\left\langle \pi d^2\underline{c}''\right\rangle_n.\nabla\rho}_{\text{neglected}}\right)+\frac{4\pi}{\rho}\int d\dot{m}f\delta d+C\left(\pi d^2\right) \tag{6.42}$$

In our present state of the art, the two terms involving $\underline{c}''$ are neglected in the NEPTUNE_CFD code. Replacing the gas density ρ by its averaged value in Eq. (6.42) yields:

$$\frac{\partial a_I}{\partial t}+\nabla.\left(a_I\overline{\overline{\underline{v}_G}}^{G}\right)=-\frac{2a_I}{3\overline{\rho_G}^{G}}\frac{D_G\overline{\rho_G}^{G}}{Dt}+\frac{4\pi}{\overline{\rho_G}^{G}}\int d\dot{m}f\delta d+C\left(\pi d^2\right) \tag{6.43}$$

Using the gas mass balance equation (3.12), the equation (6.43) can be rewritten equivalently:

$$\frac{\partial a_I}{\partial t}+\nabla.\left(a_I\overline{\overline{\underline{v}_G}}^{G}\right)=\underbrace{\frac{2a_I}{3\alpha_G}\left[\frac{\partial\alpha_G}{\partial t}+\nabla.\left(\alpha_G\overline{\overline{\underline{v}_G}}^{G}\right)\right]}_{\text{gas expansion term}}+\underbrace{\frac{4\pi}{\overline{\rho_G}^{G}}\int d\dot{m}f\delta d-\frac{2a_I}{3\alpha_G}\frac{\Gamma_G}{\overline{\rho_G}^{G}}}_{\text{phase change terms}}+\underbrace{C\left(\pi d^2\right)}_{\substack{\text{coalescence}\\ \text{and break-up}}} \tag{6.44}$$

The RHS of Eq. (6.44) contains three types of terms. The first one is called the gas expansion term (Ishii & Hibiki, 2006) and needs no further modelling. The two following terms are due to phase change and need further modelling. The last collision term includes coalescence and break-up of bubbles which also need to be modelled. In the simplest case where all the bubbles are assumed to have the Sauter mean diameter d_{32}, the phase change terms in Eq. (6.44) cancel together:

$$f(d)=\delta\left(d-d_{32}\right)\ \Rightarrow\ \frac{4\pi}{\overline{\rho_G}^{G}}\int d\dot{m}\delta\left(d-d_{32}\right)\delta d-\frac{2a_I}{3\alpha_G}\frac{\Gamma_G}{\overline{\rho_G}^{G}}=\frac{4\pi}{\overline{\rho_G}^{G}}\left(\dot{m}\left(d_{32}\right)d_{32}-\dot{m}\left(d_{32}\right)\frac{d_{32}{}^2}{d_{32}}\right)=0 \tag{6.45}$$

where the definition (4.37) of Γ_G has been used. In NEPTUNE_CFD code, we prefer to use Eq. (6.43) where the phase change term is decomposed into a wall nucleation part and another part due to phase change through the surfaces of the already existing bubbles, as we have done in the mass balance equation (6.1). Hence, Eq. (6.43) becomes:

$$\frac{\partial a_I}{\partial t} + \nabla . \left(a_I \overline{\overline{\underline{v}_G}}^G \right) = - \frac{2 a_I}{3 \overline{\rho_G}^G} \frac{D_G \overline{\rho_G}^G}{Dt} + \frac{4\pi}{\overline{\rho_G}^G} \int d \frac{q''_{LI} + q''_{VI}}{\ell} f \delta d + \frac{6 \Gamma_G^{nuc}}{\overline{\rho_G}^G d_d} + C\left(\pi d^2 \right) \quad (6.46)$$

where q''_{LI} and q''_{VI} are the heat flux densities which must be modelled in accordance to (6.26), ℓ is the latent heat of vaporisation and Γ_G^{nuc} is the gas mass production by newly nucleated bubbles at the detachment diameter d_d.

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