A SHORT INTRODUCTION TO TWO-PHASE FLOWS

Two-phase flows balance equations

Hervé Lemonnier
DM2S/STMF/LIEFT, CEA/Grenoble, 38054 Grenoble Cedex 9
Ph. +33(0)4 38 78 45 40
herve.lemonnier@cea.fr, *herve.lemonnier.sci.free.fr/TPF/TPF.htm*

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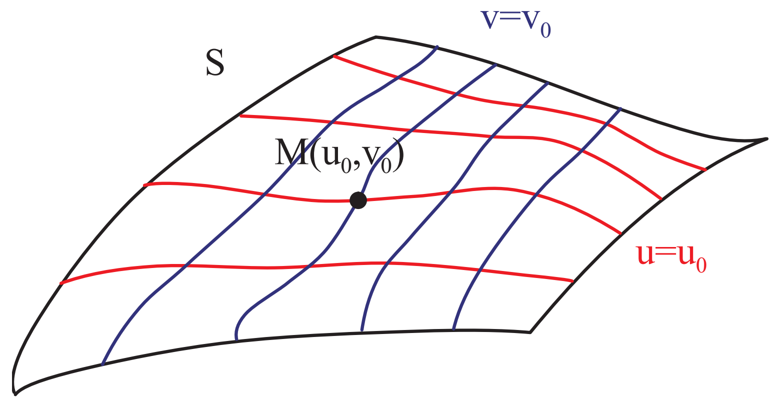
DERIVATION OF CONTINUUM MECHANICS BALANCE EQUATIONS

1. First principles (4)
   - Leibniz rule and Gauss theorem.
   - On material and arbitrary control volumes.
2. Local instantaneous balance equations (single-phase). The closure issue (I)
   - Fixed volume with an interface (discontinuity surface).
3. Local instantaneous balance for each phase and the interface (jump conditions).
   - Space averaging: 1D balance equations.
   - Time averaging: 3D local balance equations (Reynolds style).
   - Composite averaging: two-fluid model.
4. The closure issue (II)

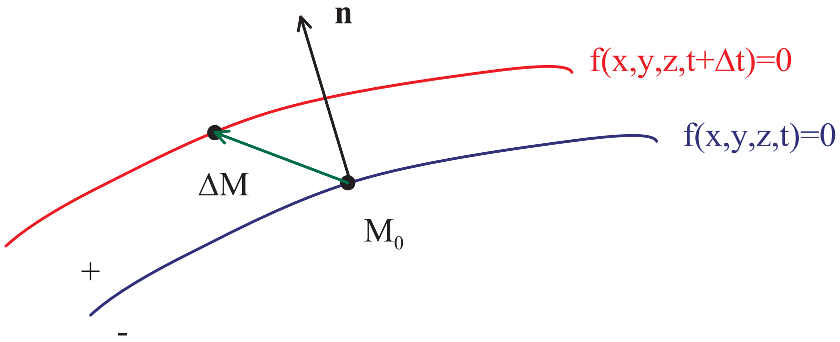
MATHEMATICAL TOOLS



- Displacement velocity of a surface, S:

$$\mathbf{v}_S \triangleq \left(\frac{\partial \mathbf{M}}{\partial t}\right)_{u,v}$$

- Depends on the choice of parameters.



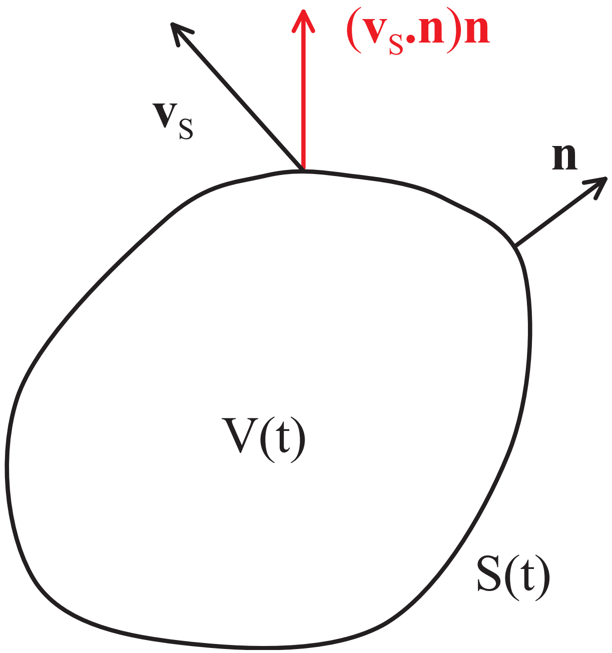
- Implicit equation: $f(x, y, z, t) \leqslant 0$ inside V

$$f(x, y, z, t + \Delta t) = f(x_0, y_0, z_0, t)$$
$$+\nabla f(M_0) \cdot \Delta M + \frac{\partial f}{\partial t}\Delta t + \cdots$$

- Geometrical displacement velocity (intrinsic, scalar):

$$\mathbf{v}_S \cdot \mathbf{n} = \lim_{\Delta t \to 0} \frac{\Delta M}{\Delta t} = -\frac{\frac{\partial f}{\partial t}}{|\nabla f|}$$

LEIBNIZ RULE

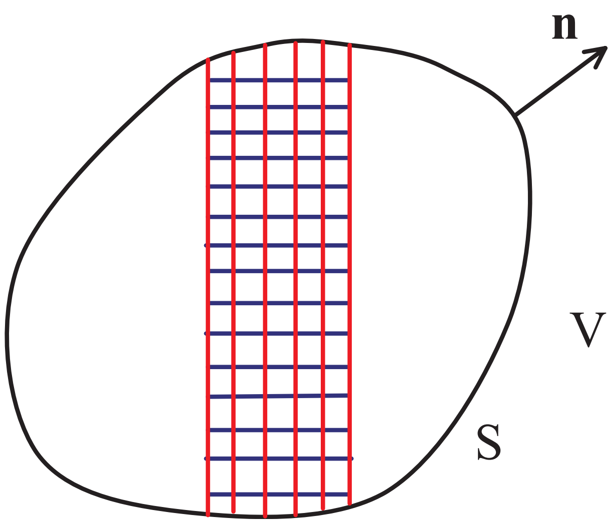


- 3D-extension of the derivation of integrals theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} f\,\mathrm{d}V = \int_{V(t)} \frac{\partial f}{\partial t}\mathrm{d}V + \int_{S(t)} f\,\mathbf{v}_S\,\boldsymbol{.}\,\mathbf{n}\,\mathrm{d}S$$

- Differential geometry theorem, S arbitrary.
- **n** points outwardly (always).
- Use: commutes time derivative and space integration.
- Material control volumes → arbitrary volumes.

GAUSS THEOREM



- Divergence is the flux per unit volume:

$$\nabla\,\boldsymbol{.}\,\mathbf{B} \triangleq \lim_{\epsilon \longrightarrow 0} \frac{1}{V_\epsilon} \int_S \mathbf{n}\,\boldsymbol{.}\,\mathbf{B}\,\mathrm{d}S$$

- Divergence theorem, Gauss-Otstrodradski (Green) :

$$\int_{V(t)} \nabla\,\boldsymbol{.}\,\mathbf{B}\,\mathrm{d}V = \int_{S(t)} \mathbf{n}\,\boldsymbol{.}\,\mathbf{B}\,\mathrm{d}S$$

- Differential geometry theorem, S and V arbitrary, **n** et ∇ on the same side. **n**, points outwards. **B**, arbitrary tensor.
- Use: some particular volume integrals $\Leftrightarrow$ surface integrals.

MATERIAL VOLUMES-ARBITRARY VOLUMES

- Let $V_m(t)$, limited by $S_m(t)$ be a material volume : $\mathbf{v}_{S_m} \cdot \mathbf{n} = \mathbf{v} \cdot \mathbf{n}$.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V_m(t)} f \,\mathrm{d}V = \int_{V_m(t)} \frac{\partial f}{\partial t} \,\mathrm{d}V + \int_{S_m(t)} f \mathbf{v} \cdot \mathbf{n} \,\mathrm{d}S$$

- Consider $V(t)$ which coincides with $V_m(t)$ at t.

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} f \,\mathrm{d}V = \int_{V(t)} \frac{\partial f}{\partial t} \,\mathrm{d}V + \int_{S(t)} f \mathbf{v}_S \cdot \mathbf{n} \,\mathrm{d}S$$

- Identity: for all $V(t)$ which coincides with $V_m(t)$ at time t,

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t} \int_{V_m(t)} f \,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} f \,\mathrm{d}V + \int_{S(t)} f (\mathbf{v} - \mathbf{v}_S) \cdot \mathbf{n} \,\mathrm{d}S}$$

A SIMPLE EXAMPLE: MASS BALANCE

- Principle: the mass of a material volume is constant.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_m(t)} \rho \,\mathrm{d}V = 0$$

- Use the identity with $f = \rho$,

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho \,\mathrm{d}V}_{\text{Mass of } V,\ m} + \underbrace{\int_{S(t)} \rho(\mathbf{v} - \mathbf{v}_S) \,.\, \mathbf{n} \,\mathrm{d}S}_{\text{Net mass flux leaving S, } M} = 0$$

- The time variation of the mass of V, m, equals the net incoming mass rate, $-M$.

$$\frac{\mathrm{d}m}{\mathrm{d}t} + M = 0, \quad \frac{\mathrm{d}m}{\mathrm{d}t} = -M$$

- First principles can be formulated on material or arbitrary volumes. Both statements are **equivalent**.

MASS BALANCE

- The time variation of the mass equals the net mass flow rate *entering* in the volume V $(\forall V)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \,\mathrm{d}V = -\int_S \rho(\mathbf{v} - \mathbf{v}_S)\,\mathbf{.}\,\mathbf{n}\,\mathrm{d}S. \tag{1}$$

- Particular cases,
  - For a fixed volume, $\mathbf{v}_S \,\mathbf{.}\, \mathbf{n} = 0$,
  - For a material volume, $\mathbf{v}_S \,\mathbf{.}\, \mathbf{n} = \mathbf{v} \,\mathbf{.}\, \mathbf{n}$

SPECIES BALANCE

- The time variation of the mass of component α equals (i) the net incoming mass rate of α and (ii) the production in the volume V $(\forall V)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho_\alpha \,\mathrm{d}V = -\int_S \rho_\alpha(\mathbf{v}_\alpha - \mathbf{v}_S)\cdot\mathbf{n}\,\mathrm{d}V + \int_V r_\alpha \,\mathrm{d}V$$

- Add all equations for α gives the mixture mass balance.
- $\sum_\alpha r_\alpha = 0$.
- Chemicals redistribution, no overall net mass production.

LINEAR MOMENTUM BALANCE

- The time variation of the linear momentum equals the sum of (i) the incoming momentum flux, (ii) the applied forces ($\forall V$).

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{v}\,\mathrm{d}V = -\int_S \rho\mathbf{v}(\mathbf{v}-\mathbf{v}_S)\,\boldsymbol{.}\,\mathbf{n}\,\mathrm{d}S + \int_S \mathbf{n}\,\boldsymbol{.}\,\mathbb{T}\,\mathrm{d}S + \int_V \rho\mathbf{g}\,\mathrm{d}V \qquad (2)$$

- $\mathbb{T}$: stress tensor, contact forces.
- $\mathbf{g}$: volume forces.
- NB: vector equation.

ANGULAR MOMENTUM BALANCE

- The time variation of the momentum moment equals the sum of (i) the net incoming flux of moment of momentum and (ii) the applied torques ($\forall V$).

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\mathbf{r}\times\mathbf{v}\,\mathrm{d}V = -\int_S \rho\mathbf{r}\times\mathbf{v}(\mathbf{v}-\mathbf{v}_S)\,\boldsymbol{.}\,\mathbf{n}\,\mathrm{d}S + \int_S \mathbf{r}\times(\mathbf{n}\,\boldsymbol{.}\,\mathbb{T})\,\mathrm{d}S + \int_V \mathbf{r}\times\rho\mathbf{g}\,\mathrm{d}V \tag{3}$$

- When torques results only of applied forces (non polar fluids). Take two get the third.
  - The stress tensor is symmetrical.
  - The linear momentum balance is satisfied.
  - The angular momentum balance is satisfied.

TOTAL ENERGY BALANCE

- Equivalent to the first principle of thermodynamics: the time variation of the total energy of a closed system equals the sum of (i) the thermal power added and (ii) the power of external forces applied to the system.
- The time variation of the total energy (internal and mechanical) equals the sum of (i) the incoming total energy flux, (ii) the mechanical power of the applied forces and (iii) the thermal power given to the system.($\forall V$).

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\left(u+\frac{1}{2}\mathrm{v}^2\right)\mathrm{d}V = -\int_S \rho\left(u+\frac{1}{2}\mathrm{v}^2\right)(\mathbf{v}-\mathbf{v}_S)\,\boldsymbol{.}\,\mathbf{n}\,\mathrm{d}S + \int_S (\mathbf{n}\,\boldsymbol{.}\,\mathbb{T})\,\boldsymbol{.}\,\mathbf{v}\,\mathrm{d}S + \int_V \rho\mathbf{g}\,\boldsymbol{.}\,\mathbf{v}\,\mathrm{d}V - \int_S \mathbf{q}\,\boldsymbol{.}\,\mathbf{n}\,\mathrm{d}S + \int_V q'''\,\mathrm{d}V \tag{4}$$

- q''': volume heat sources (Joulean heating, radiation absorption, *etc.*) NOT of thermodynamical origin, heat of reaction, phase transition of any order...
- The process is arbitrary: reversible or not.

ENTROPY BALANCE AND SECOND PRINCIPLE

- The time variation of the entropy of a closed and isolated system is non negative.
- The time variation of the entropy equals (i) the net inflow of entropy, (ii) the entropy given to the system in a reversible manner, (iii) the entropy sources ($\forall V$).

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho s \,\mathrm{d}V = -\int_S \rho s(\mathbf{v}-\mathbf{v}_S)\,\boldsymbol{.}\,\mathbf{n}\,\mathrm{d}S - \int_S \mathbf{n}\,\boldsymbol{.}\,\mathbf{j}_s\,\mathrm{d}S + \int_V \frac{q'''}{T}\,\mathrm{d}V + \int_V \sigma\,\mathrm{d}V, \tag{5}$$

$$\sigma \geqslant 0.$$

- The second principle is "only" $\sigma \geqslant 0$.
- When reversible, $\sigma = 0$.

GENERALIZED BALANCE EQUATION

Balance equations have similar forms,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\psi\,\mathrm{d}V = -\int_S \mathbf{n}\,\boldsymbol{.}\,\rho(\mathbf{v}-\mathbf{v}_S)\psi\,\mathrm{d}S - \int_S \mathbf{n}\,\boldsymbol{.}\,\mathbf{j}_\psi\,\mathrm{d}S + \int_V \phi_\psi \mathrm{d}V.$$

Balance	ψ	$\mathbf{j}_\psi$	ϕ_ψ
Mass	1		
Species α	ω_α	$\mathbf{j}_\alpha$	r_α
L. momentum	$\mathbf{v}$	$-\mathbb{T}$	$\rho\mathbf{g}$
A. momentum	$\mathbf{r}\times\mathbf{v}$	$-\mathbb{T}\,\boldsymbol{.}\,\mathbb{R}^{(*)}$	$\mathbf{r}\times\rho\mathbf{g}$
Total energy	$u+\frac{1}{2}\mathrm{v}^2$	$\mathbf{q}-\mathbb{T}\,\boldsymbol{.}\,\mathbf{v}$	$\rho\mathbf{g}\,\boldsymbol{.}\,\mathbf{v}+q'''$
Entropy	s	$\mathbf{j}_s$	$\sigma+\frac{q'''}{T}$

$(*)\mathbb{R},\ R_{ij}=\epsilon_{ijk}r_k$

PRIMARY LOCAL EQUATIONS

Leibniz rule,

$$\int_V \frac{\partial \rho\psi}{\partial t}\,\mathrm{d}V = -\int_S \mathbf{n}\,\boldsymbol{\cdot}\,\rho\mathbf{v}\psi\,\mathrm{d}S - \int_S \mathbf{n}\,\boldsymbol{\cdot}\,\mathbf{j}_\psi\,\mathrm{d}S + \int_V \phi_\psi \mathrm{d}V.$$

Gauss theorem, $\forall V \subset D_f$,

$$\int_V \left[\frac{\partial \rho\psi}{\partial t} + \nabla\,\boldsymbol{\cdot}\,(\rho\mathbf{v}\psi) + \nabla\,\boldsymbol{\cdot}\,\mathbf{j}_\psi - \phi_\psi\right]\mathrm{d}V = 0$$

Instantaneous local *primary* balances,

$$\boxed{\frac{\partial \rho\psi}{\partial t} = \underbrace{-\nabla\,\boldsymbol{\cdot}\,(\rho\mathbf{v}\psi)}_{\text{Convection}}\;\underbrace{-\nabla\,\boldsymbol{\cdot}\,\mathbf{j}_\psi}_{\text{Diffusion}}\;\underbrace{+\phi_\psi}_{\text{Source}}}$$

Balance on a fixed and infinitesimal volume, **strictly equivalent** to first principles.

TOTAL FLUX FORM

Total flux form (Bird *et al.* , 2007), stationary flows.

$$\frac{\partial \rho \psi}{\partial t} = -\nabla \cdot \mathbf{j}^t_\psi + \phi_\psi$$

Balance	total flux $\mathbf{j}^t_\psi$	convective flux $\rho\psi\mathbf{v}$	diffusive flux $\mathbf{j}_\psi$
Mass	$\mathbf{n} =$	$\rho\mathbf{v}$	
Species	$\mathbf{n}_\alpha =$	$\rho\omega_\alpha\mathbf{v}$	$\mathbf{j}_\alpha$
Momentum	$\phi =$	$\rho\mathbf{v}\mathbf{v}$	$-\mathbb{T}$
Total energy	$\mathbf{e} =$	$\rho\mathbf{v}\left(u + \frac{1}{2}v^2\right)$	$\mathbf{q} - \mathbb{T} \cdot \mathbf{v}$
Entropy	$\mathbf{j}^t_s =$	$\rho s\mathbf{v}$	$\mathbf{j}_s$

NB: Some authors may use different sign conventions for fluxes. Don't pick up an equation from a text without care...

CONVECTIVE FORM

Combine with the mass balance,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v})$$

Expand products in the balance equation,

$$\frac{\partial \rho \psi}{\partial t} = -\nabla \cdot (\rho \mathbf{v} \psi) - \nabla \cdot \mathbf{j}_\psi + \phi_\psi$$

$$\rho \frac{\partial \psi}{\partial t} + \psi \frac{\partial \rho}{\partial t} = -\psi \nabla \cdot (\rho \mathbf{v}) - \rho \mathbf{v} \cdot \nabla \psi - \nabla \cdot \mathbf{j}_\psi + \phi_\psi$$

Definition of the convective derivative:

$$\frac{\mathrm{D} f}{\mathrm{D} t} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f$$

$$\boxed{\rho \frac{\mathrm{D} \psi}{\mathrm{D} t} = -\nabla \cdot \mathbf{j}_\psi + \phi_\psi}$$

Balance on a material volume (infinitesimal). Only diffusive fluxes. Practical form to derive secondary equations.

SUMMARY OF CONTINUUM MECHANICS EQUATIONS

For a pure fluid, on an arbitrary control volume,

- Mass balance (1)
- Linear momentum balance (2)
- Angular momentum balance (3)
- Total energy balance (4)
- Entropy inequality (5)

Local primary balance equations,

(1)→ Mass balance (6)

(2)→ Momentum balance (7)

(3)→ Stress tensor symmetry

(4)→ Total energy balance (8)

(5)→ Entropy inequality (9)

CONTINUUM MECHANICS EQUATIONS

Secondary balance equations, for a pure fluid,

- Mechanical energy balance, (10) $\mathbf{.v}$ momentum balance.
- Internal energy balance (11), total energy balance (8)-(10).
- Enthalpy balance (12). (11), $h \triangleq u + p/\rho$
- Entropy balance (13), (11), $\mathrm{d}u = T\mathrm{d}s - p\mathrm{d}v$ (Gibbs).
- Comparing to entropy inequality (9), provides $\mathbf{j}_s$ and σ.

THE CLOSURE ISSUE (I)

- In balance equations,
  - Local variables, $\mathbf{v}$, $\mathbf{v}_\alpha$, p, u, etc.
  - Unknown fluxes, $\mathbf{j}_\alpha$, $\mathbb{T}$, $\mathbf{q}$, $\mathbf{j}_s$. NB: $\mathbb{T} = -p\mathbb{I} + \mathbb{V}$
  - Unknown sources, r_α, σ.
- First principles cannot provide expressions for fluxes.

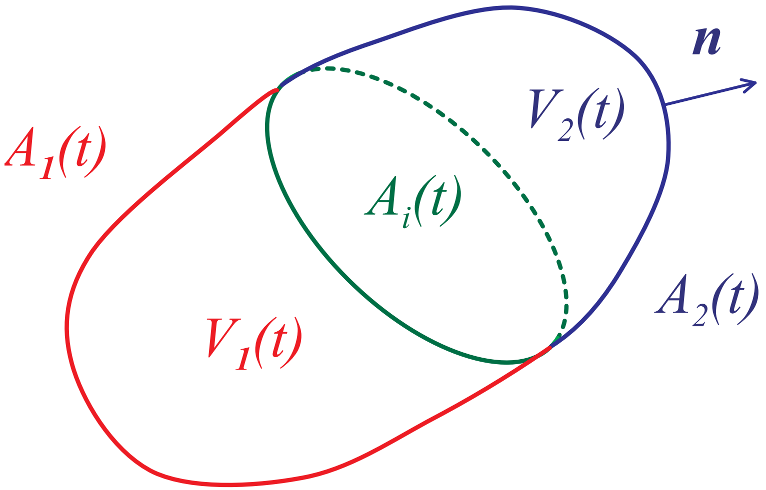
The CME are not closed.

- An extended interpretation of the second principle,
  - provides entropy sources. For a pure fluid, $T\sigma = \mathbf{q} \cdot \nabla T + \mathbb{V} : \nabla \mathbf{v}$.
  - provides the thermodynamic equilibrium conditions, $\sigma = 0$,
  - provides constraints on possible closure to ensure return to equilibrium. Linearity assumption, transport properties,

$$\mathbb{T} = \mu(\nabla \mathbf{v} + \mathbf{v}\nabla) + (\zeta - \frac{2}{3}\mu)\nabla \cdot \mathbf{v}, \quad \mathbf{q} = -\kappa \nabla T, \quad \mu,\ \zeta, \kappa \geqslant 0$$

- Transport properties must be measured or modeled beyond CME scope.

TWO-PHASE LOCAL BALANCE EQUATIONS



- Example: mass balance, $V = V_1 \cup V_2$, $A = A_1 \cup A_2$ fixed. Interfaces,surface of discontinuity.

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \, \mathrm{d}V = -\int_A \rho \mathbf{v} \,\boldsymbol{.}\, \mathbf{n} \, \mathrm{d}S, \quad \forall V$$

- Split contributions from V_1 and V_2:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_1} \rho \, \mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\int_{V_2} \rho \, \mathrm{d}V = -\int_{A_1} \rho \mathbf{v} \,\boldsymbol{.}\, \mathbf{n} \, \mathrm{d}S - \int_{A_2} \rho \mathbf{v} \,\boldsymbol{.}\, \mathbf{n} \, \mathrm{d}S$$

- For $V_1(t)$ (not fixed), Leibniz rule:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_1} \rho_1 \, \mathrm{d}V = \int_{V_1} \frac{\partial \rho_1}{\partial t} \, \mathrm{d}V + \int_{A_i(t)} \rho_1 \mathbf{v}_{A_i} \,\boldsymbol{.}\, \mathbf{n}_1 \, \mathrm{d}A$$

- Gauss theorem:

$$\int_{A_1} \rho \mathbf{v}_1 \,\boldsymbol{.}\, \mathbf{n}_1 \, \mathrm{d}S = \int_{V_1} \nabla \,\boldsymbol{.}\, (\rho_1 \mathbf{v}_1) \, \mathrm{d}S - \int_{A_i} \rho_1 \mathbf{v}_1 \,\boldsymbol{.}\, \mathbf{n} \, \mathrm{d}A$$

- Apply the same procedure for V_2, sum up all contributions,

TWO-PHASE MASS BALANCE

- Collect integral terms wrt dimension, $\forall V$,

$$\sum_{k=1}^{2} \int_{V_k} \left(\frac{\partial \rho_k}{\partial t} + \nabla \cdot (\rho_k \mathbf{v}_k) \right) \mathrm{d}V - \int_{A_i} (\rho_1(\mathbf{v}_1 - \mathbf{v}_i) + \rho_2(\mathbf{v}_2 - \mathbf{v}_i)) \, \mathrm{d}A = 0$$

- Local mass balance, $k = 1, 2$, for all points in V_k (PDE),

$$\frac{\partial \rho_k}{\partial t} + \nabla \cdot (\rho_k \mathbf{v}_k) = 0$$

- For all points of the interface, jump condition,

$$\underbrace{\rho_1(\mathbf{v}_1 - \mathbf{v}_i) \cdot \mathbf{n}_1}_{\dot{m}_1} + \underbrace{\rho_2(\mathbf{v}_2 - \mathbf{v}_i) \cdot \mathbf{n}_2}_{\dot{m}_2} = 0$$

- Jump condition is the mass balance of the interface, $\dot{m}_k = \rho_k(\mathbf{v}_k - \mathbf{v}_i) \cdot \mathbf{n}_k$.

LOCAL BALANCE EQUATIONS

- Use the generalized local balance equation, same procedure, $\forall V$

$$\sum_{k=1}^{2} \int_{V_k} \left(\frac{\partial \rho_k \psi_k}{\partial t} + \nabla \cdot (\rho_k \psi_k \mathbf{v}_k) + \nabla \cdot (\mathbf{j}_{\psi k}) - \phi_k \right) \mathrm{d}V$$

$$- \int_{A_i} \sum_{k=1}^{2} (\dot{m}_k \psi_k + \mathbf{n}_k \cdot \mathbf{j}_{\psi k} + \phi_i) \, \mathrm{d}A = 0$$

- At every points of each phase,

$$\frac{\partial \rho_k \psi_k}{\partial t} + \nabla \cdot (\rho_k \psi_k \mathbf{v}_k) + \nabla \cdot (\mathbf{j}_{\psi k}) - \phi_k = 0$$

- At every points of the interface, jump condition, balance of the interface.

$$\sum_{k=1}^{2} (\dot{m}_k \psi_k + \mathbf{n}_k \cdot \mathbf{j}_{\psi k} + \phi_i) = 0$$

- ϕ_i: entropy source at the interface.

JUMP CONDITIONS

- Mass balance,

$$\rho_1(\mathbf{v}_1 - \mathbf{v}_i) \centerdot \mathbf{n}_1 + \rho_2(\mathbf{v}_2 - \mathbf{v}_i) \centerdot \mathbf{n}_2 = 0$$

$$\dot{m}_1 + \dot{m}_2 = 0$$

- No phase change: $\dot{m}_k = 0$,

$$\dot{m}_1 = \dot{m}_2 = 0$$

- Assumption: no slip at the interface ($\phi_i = 0$),

$$(\mathbf{v}_1 - \mathbf{v}_i) \centerdot \mathbf{n}_1 = 0, \quad (\mathbf{v}_2 - \mathbf{v}_i) \centerdot \mathbf{n}_2 = 0$$

$$(\mathbf{v}_1 - \mathbf{v}_2) \centerdot \mathbf{n}_1 = 0 \Rightarrow \mathbf{v}_1 = \mathbf{v}_2$$

JUMP CONDITIONS (CT'D)

- Momentum balance,

$$\dot{m}_1 \mathbf{v}_1 + \dot{m}_2 \mathbf{v}_2 - \mathbf{n}_1 \,\mathbf{.}\, \mathbb{T}_1 - \mathbf{n}_2 \,\mathbf{.}\, \mathbb{T}_2 = 0$$

- When no viscosity, $\mathbb{T} = -p\mathbb{I} + \mathbb{V}$, $\mathbf{v} = \mathbf{v}^t + \mathbf{v}^n$, $\mathbf{v}^n = \mathbf{n}(\mathbf{v} \,\mathbf{.}\, \mathbf{n})$,

$$\begin{cases} \dot{m}_1(\mathbf{v}_1^n - \mathbf{v}_2^n) + (p_1 - p_2)\mathbf{n}_1 = 0 \\ \mathbf{v}_1^t = \mathbf{v}_2^t \end{cases}$$

- General case,

$$\dot{m}_1(\mathbf{v}_1 - \mathbf{v}_2) + (p_1 - p_2)\mathbf{n}_1 - \mathbf{n}_1 \,\mathbf{.}\, (\mathbb{V}_1 - \mathbb{V}_2) = 0$$

JUMP CONDITIONS (CT'D)

- Particular case: 1D flow, $\mathbf{v}_k(x) \perp$ interface

$$\perp : \quad \dot{m}_1(\mathbf{v}_1 - \mathbf{v}_2) \cdot \mathbf{n}_1 + (p_1 - p_2) - \mathbf{n}_1 \cdot (\mathbb{V}_1 - \mathbb{V}_2) \cdot \mathbf{n}_1 = 0$$

- 1D incompressible flow, $\frac{\mathrm{d}v_k}{\mathrm{d}x} = 0 \Rightarrow \mathbb{V}_k = 0$,

$$\dot{m}_1(\mathbf{v}_1 - \mathbf{v}_2) \cdot \mathbf{n}_1 + (p_1 - p_2) = 0$$

- From the mass balance, definition: $\dot{m}_k = \rho_k(\mathbf{v}_k - \mathbf{v}_i) \cdot \mathbf{n}_k$,

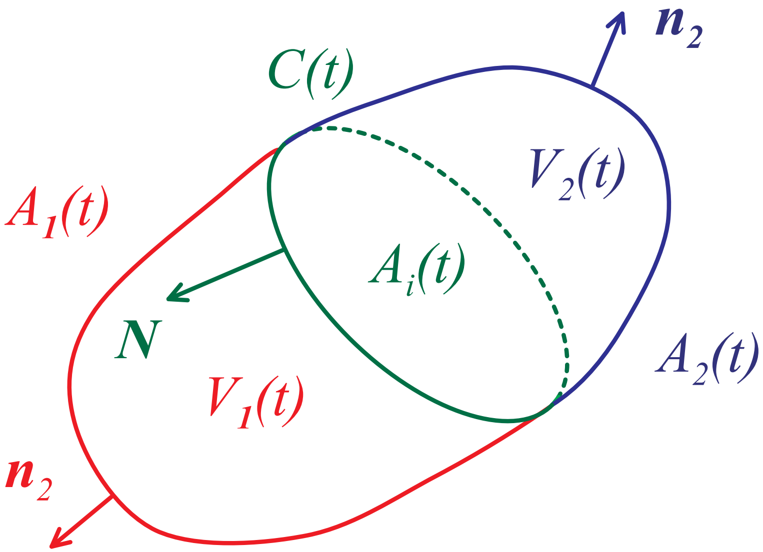
$$\dot{m}_1 \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) = (\mathbf{v}_1 - \mathbf{v}_2) \cdot \mathbf{n}_1$$

- Results, pressure jump, recoil force,

$$p_1 - p_2 = \frac{\rho_1 - \rho_2}{\rho_1 \rho_2} \dot{m}_1^2.$$

- $p_1 - p_2 \propto \rho_1 - \rho_2$ whatever $\dot{m}_1$.

MOMENTUM BALANCE AND SURFACE TENSION



- Momentum balance on a fixed volume. Forces:

$$= \int_{C(t)} \sigma \mathbf{N} \, \mathrm{d}l + \int_{A_1 \cup A_2} \mathbf{n}_k \,.\, \mathbb{T}_k \, \mathrm{d}S + \int_{V_1 \cup V_2} \rho_k \mathbf{F}_k \, \mathrm{d}V.$$

- Théorème de Gauss Aris (1962), Delhaye (1974) :

$$\int_{C(t)} \sigma \mathbf{N} \, \mathrm{d}l = \int_{A_i(t)} (\nabla_S \sigma - \mathbf{n} \sigma \nabla_S \,.\, \mathbf{n}) \, \mathrm{d}S$$

- ∇_S: surface gradient, $\nabla_S \,.$: surface divergence. Momentum balance interface:

$$\dot{m}_1 \mathbf{v}_1 + \dot{m}_2 \mathbf{v}_2 - \mathbf{n}_1 \,.\, \mathbb{T}_1 - \mathbf{n}_2 \,.\, \mathbb{T}_2 = -\nabla_S \sigma + \mathbf{n} \sigma \nabla_S \,.\, \mathbf{n}$$

- $\nabla_S \sigma$: Marangoni force, $\mathbf{n} \sigma \nabla_S \,.\, \mathbf{n}$: capillary pressure, Laplace pressure jump.

$$\mathbf{n} \sigma \nabla_S \,.\, \mathbf{n} = 2 H \mathbf{n}$$

- H: mean curvature of the surface. Circular cylinder: 1/R, sphere: 2/R.

EXAMPLE: 2D INTERFACES

- Momentum jump at the interface,

$$\dot{m}_1 \mathbf{v}_1 + \dot{m}_2 \mathbf{v}_2 - \mathbf{n}_1 \,.\, \mathbb{T}_1 - \mathbf{n}_2 \,.\, \mathbb{T}_2 + \frac{\mathrm{d}\sigma}{\mathrm{d}l} \tau - \frac{\sigma}{R} \mathbf{n} = 0$$

- For a non viscous fluid, $\mathbb{T} = -p\mathbb{I} + \mathbb{V}$,no phase change,

$$\mathbf{n}_1 (p_1 - p_2) + \frac{\mathrm{d}\sigma}{\mathrm{d}l} \tau - \frac{\sigma}{R} \mathbf{n} = 0$$

- Laplace relation, $\perp$: $(p_1 - p_2) = \dfrac{\sigma}{R} \mathbf{n}_1 \,.\, \mathbf{n}$
- Inconsistency, // : $\mu_k = 0 \Rightarrow \dfrac{\mathrm{d}\sigma}{\mathrm{d}l} = 0$
- Marangoni effect for viscous fluids, $\sigma(T)$, $\sigma(c)$,

$$-(\mathbf{n}_1 \,.\, \mathbb{V}_1 + \mathbf{n}_2 \,.\, \mathbb{V}_2) \,.\, \tau + \frac{\mathrm{d}\sigma}{\mathrm{d}l} = 0$$

- Be careful with the parameter selection. Pressure is always higher in the concavity side (balloon).

JUMP CONDITIONS (CT'D)

- Total energy balance:

$$\dot{m}_1\left(u_1+\frac{1}{2}v_1^2\right)+\dot{m}_2\left(u_2+\frac{1}{2}v_2^2\right)+\mathbf{q}_1\mathbin{.}\mathbf{n}_1+\mathbf{q}_2\mathbin{.}\mathbf{n}_2-\mathbf{n}_1\mathbin{.}\mathbb{T}_1\mathbin{.}\mathbf{v}_1-\mathbf{n}_2\mathbin{.}\mathbb{T}_2\mathbin{.}\mathbf{v}_2=0$$

- Enthalpy form, 3 common assumptions,
  - phase change is the dominant effect,
  - variation of mechanical energy can be neglected,
  - the effect of pressure and viscous stress jump can be neglected (no surface tension),

$$\dot{m}_1h_1+\dot{m}_2h_2+\mathbf{q}_1\mathbin{.}\mathbf{n}_1+\mathbf{q}_2\mathbin{.}\mathbf{n}_2=0$$

- More on the derivation, see Delhaye (1974, 2008).
- Thermodynamic equilibrium condition at the interface:

$$\mathbf{v}_1^t=\mathbf{v}_2^t,\quad T_1=T_2,\quad g_1-g_2=\frac{1}{2}\dot{m}_1^2\left(\frac{1}{\rho_2^2}-\frac{1}{\rho_1^2}\right)-\left(\frac{\mathbf{n}_2\mathbin{.}\mathbb{V}_2\mathbin{.}\mathbf{n}_2}{\rho_2}-\frac{\mathbf{n}_1\mathbin{.}\mathbb{V}_1\mathbin{.}\mathbf{n}_1}{\rho_1}\right)$$

USE OF LOCAL EQUATIONS

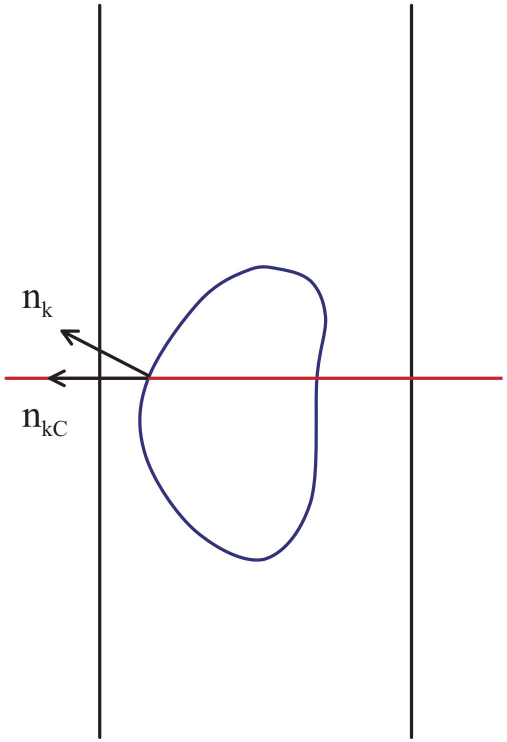
- First principles→ balance on arbitrary control volumes
  - Local phase equations,
  - Jump conditions at the interface (see also the Rankine-Hugoniot eqs).
- Flows with simple interface configuration
  - Stability of a liquid film,
  - Growth/collapse of a vapor bubble (nucleate boiling, cavitation).
- More general problems,
  - Tremendously large number of interfaces, non-equilibrium.
  - Large scale fluctuations, intermittency, engineers seek for mean values.

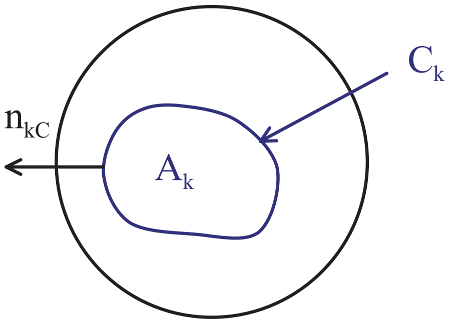
⇒ Space averaging of local equations (area-averaged): 1D models

⇒ Time averaging of local equations: CMFD (3D codes)

⇒ Space and time averaging, composite averaging: two-fluid 1D model, 1D codes, system codes.

AREA-AVERAGED BALANCE EQUATIONS





- Area-averaging operator:

$$< f_k >_2 = \frac{1}{A_k} \int_{A_k} f_k \, \mathrm{d}A$$

- How to get a balance equation for a mean value?
  Average the local balance on A_k. Example, mass balance,

$$\frac{\partial \rho_k}{\partial t} + \nabla \cdot (\rho_k \mathbf{v}_k) = 0$$

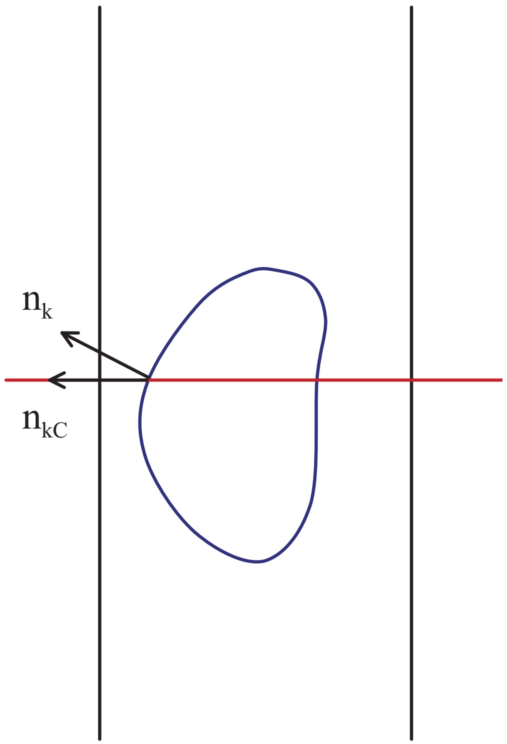
- Integrate on A_k,

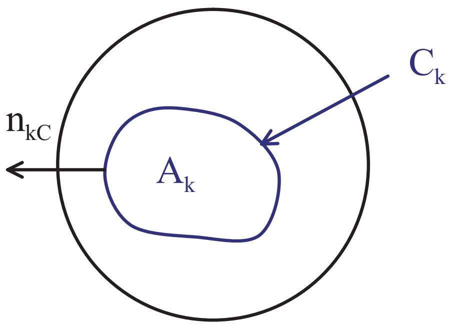
$$\int_{A_k} \frac{\partial \rho_k}{\partial t} \, \mathrm{d}A + \int_{A_k} \nabla \cdot (\rho_k \mathbf{v}_k) \, \mathrm{d}A = 0$$

- Limiting forms of the Leibniz rule and Gauss theorems,

$$\frac{\partial}{\partial t} \underbrace{\int_{A_k} \rho_k \, \mathrm{d}A}_{A_k < \rho_k >_2} + \cdots + \frac{\partial}{\partial z} \underbrace{\int_{A_k} \rho_k w_k \, \mathrm{d}A}_{A_k < \rho_k w_k >_2} + \cdots = 0$$

MATHEMATICAL TOOLS





- Limiting form of the Leibniz rule,

$$\frac{\partial}{\partial t}\int_{A_k} f_k \mathrm{d}A = \int_{A_k} \frac{\partial f_k}{\partial t}\mathrm{d}A + \int_{C_k} f_k \mathbf{v}_i \cdot \mathbf{n}_k \frac{\mathrm{d}l}{\mathbf{n}_k \cdot \mathbf{n}_{kC}}$$

- Limiting form of the Gauss theorem,

$$\int_{A_k} \nabla \cdot \mathbf{B}\mathrm{d}A = \frac{\partial}{\partial z}\int_{A_k} \mathbf{n}_z \cdot \mathbf{B}\mathrm{d}A + \int_{C_k} \mathbf{n}_k \cdot \mathbf{B}\frac{\mathrm{d}l}{\mathbf{n}_k \cdot \mathbf{n}_{kC}}$$

- First useful identity, $\mathbf{B} = \mathbf{n}_z$

$$\frac{\partial A_k}{\partial z} = -\int_{C_k} \mathbf{n}_k \cdot \mathbf{n}_z \frac{\mathrm{d}l}{\mathbf{n}_k \cdot \mathbf{n}_{kC}} \tag{1}$$

- Second useful identity, $\mathbb{B} = p\mathbb{I}$

$$\int_{A_k} \nabla p \,\mathrm{d}A = \frac{\partial}{\partial z}\int_{A_k} p\mathbf{n}_z \mathrm{d}A + \int_{C_k} p\mathbf{n}_k \frac{\mathrm{d}l}{\mathbf{n}_k \cdot \mathbf{n}_{kC}} \tag{2}$$

AREA-AVERAGED BALANCE EQUATIONS (CT'D)

- Integrate on A_k,

$$\int_{A_k} \frac{\partial \rho_k}{\partial t} \, \mathrm{d}A + \int_{A_k} \nabla \cdot (\rho_k \mathbf{v}_k) \, \mathrm{d}A = 0$$

- Leibniz rule and Gauss theorem,

$$\frac{\partial}{\partial t} A_k < \rho_k >_2 + \frac{\partial}{\partial z} A_k < \rho_k w_k >_2 = - \int_{C_k} \dot{m}_k \frac{\mathrm{d}l}{\mathbf{n}_k \cdot \mathbf{n}_{kC}}$$

- Γ_k: production rate of phase k [kg/s/m] per unit length of pipe.

$$\Gamma_k = - \int_{C_k} \dot{m}_k \frac{\mathrm{d}l}{\mathbf{n}_k \cdot \mathbf{n}_{kC}}$$

- No phase change: $\dot{m}_k = 0 \Rightarrow \Gamma_k = 0$.
- Mass balance of the interface, $\dot{m}_1 + \dot{m}_2 = 0 \Rightarrow \Gamma_1 + \Gamma_2 \equiv 0$.
- The area-averaged mass balance is **not closed**.

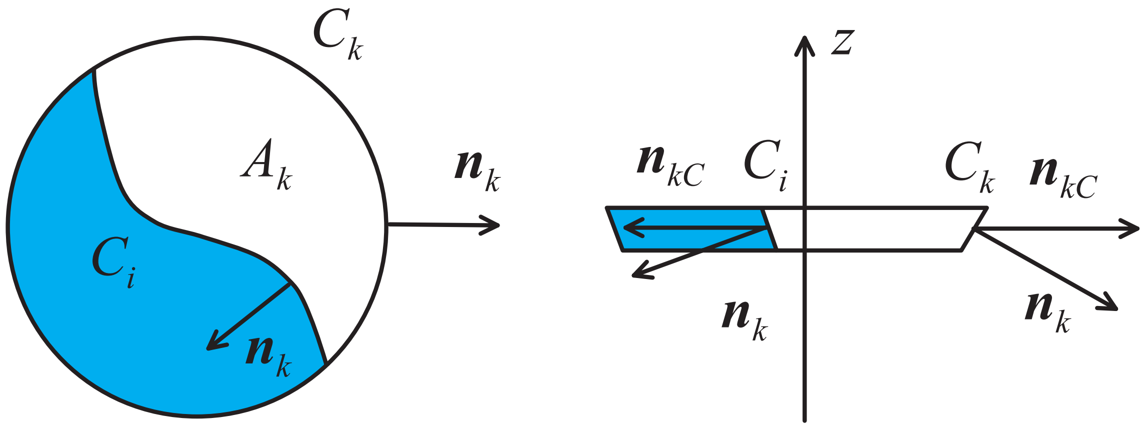
AREA-AVERAGED BALANCE EQUATIONS

- Based on the general form of the local balance equations.,

$$\frac{\partial}{\partial t} A_k < \rho_k \psi_k >_2 + \frac{\partial}{\partial z} A_k < \mathbf{n}_z \cdot \rho_k \mathbf{v}_k \psi_k >_2 + \frac{\partial}{\partial z} A_k < \mathbf{n}_z \cdot \mathbf{j}_{\psi k} >_2 - A_k < \phi_k >_2$$

$$= - \int_{C_i} (\dot{m}_k \psi_k + \mathbf{n}_k \cdot \mathbf{j}_{\psi k}) \frac{\mathrm{d}l}{\mathbf{n}_k \cdot \mathbf{n}_{kC}} - \int_{C_k} \mathbf{n}_k \cdot \mathbf{j}_{\psi k} \frac{\mathrm{d}l}{\mathbf{n}_k \cdot \mathbf{n}_{kC}}$$

- $\delta A_k = C_i \cup C_k$, C_k wall fraction wetted by k, C_i, interface.



MOMENTUM BALANCE

- Note on the momentum balance, *vector* equation,

$$\frac{\partial}{\partial t} A_k < \rho_k \mathbf{v}_k >_2 + \frac{\partial}{\partial z} A_k < \rho_k w_k \mathbf{v}_k >_2 - \frac{\partial}{\partial z} A_k < \mathbf{n}_z \centerdot \mathbb{T}_k >_2 - A_k < \rho_k \mathbf{g}_k >_2$$

$$= - \int_{C_i} \left(\dot{m}_k \mathbf{v}_k - \mathbf{n}_k \centerdot \mathbb{T}_k \right) \frac{\mathrm{d}l}{\mathbf{n}_k \centerdot \mathbf{n}_{kC}} + \int_{C_k} \mathbf{n}_k \centerdot \mathbb{T}_k \frac{\mathrm{d}l}{\mathbf{n}_k \centerdot \mathbf{n}_{kC}}$$

- Projection on $\mathbf{n}_z$: right dot product, $w_k = \mathbf{v}_k \centerdot \mathbf{n}_z$, stress tensor decomposition,

$$\frac{\partial}{\partial t} A_k < \rho_k w_k >_2 + \frac{\partial}{\partial z} A_k < \rho_k w_k^2 >_2 + \frac{\partial}{\partial z} A_k < p_k >_2 - \frac{\partial}{\partial z} A_k < \mathbf{n}_z \centerdot \mathbb{V}_k \centerdot \mathbf{n}_z >_2$$

$$- A_k < \rho_k g_z >_2 = - \int_{C_i} \left(\dot{m}_k w_k - \mathbf{n}_k \centerdot \mathbb{T}_k \centerdot \mathbf{n}_z \right) \frac{\mathrm{d}l}{\mathbf{n}_k \centerdot \mathbf{n}_{kC}} + \int_{C_k} \mathbf{n}_k \centerdot \mathbb{T}_k \centerdot \mathbf{n}_z \frac{\mathrm{d}l}{\mathbf{n}_k \centerdot \mathbf{n}_{kC}}$$

- Identity (1), assume $< p_k >_2 = p_C$,

$$- \int_{C_k \cup C_i} p_k \mathbf{n}_k \centerdot \mathbf{n}_z \frac{\mathrm{d}l}{\mathbf{n}_k \centerdot \mathbf{n}_{kC}} = - p_C \int_{C_k \cup C_i} \mathbf{n}_k \centerdot \mathbf{n}_z \frac{\mathrm{d}l}{\mathbf{n}_k \centerdot \mathbf{n}_{kC}} = < p_k >_2 \frac{\partial A_k}{\partial z}$$

- Other choices are possible, introduce an excess pressure, p_i, $p_C = < p_k >_2 + p_i \cdots$

MOMENTUM BALANCE (CT'D)

- Momentum balance, single-pressure, $< p_k >_2 = p_C$,

$$\frac{\partial}{\partial t} A_k < \rho_k w_k >_2 + \frac{\partial}{\partial z} A_k < \rho_k w_k^2 >_2 + A_k \frac{\partial}{\partial z} < p_k >_2 - \frac{\partial}{\partial z} A_k < \mathbf{n}_z \centerdot \mathbb{V}_k \centerdot \mathbf{n}_z >_2$$

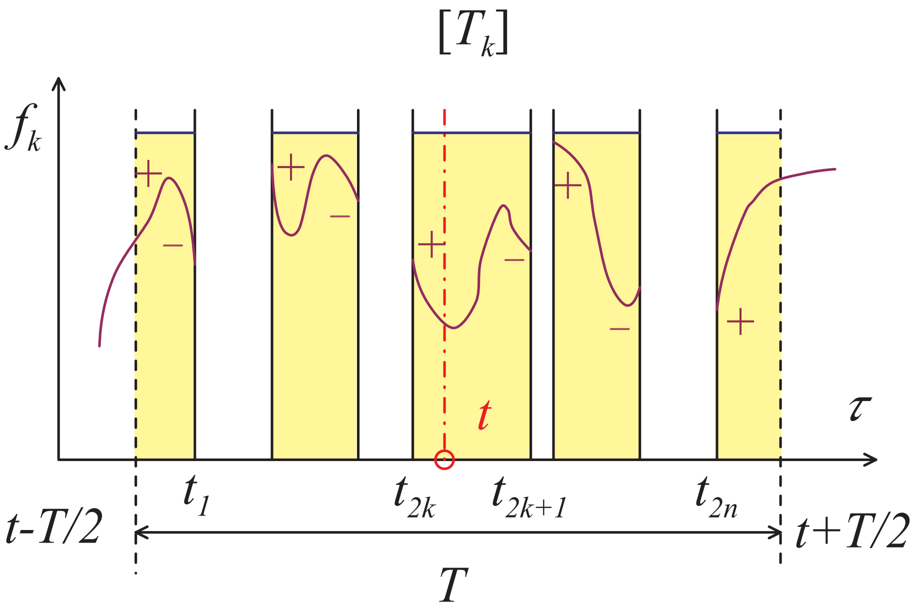
$$-A_k < \rho_k g_z >_2 = -\int_{C_i} \left(\dot{m}_k w_k - \mathbf{n}_k \centerdot \mathbb{V}_k \centerdot \mathbf{n}_z \right) \frac{\mathrm{d}l}{\mathbf{n}_k \centerdot \mathbf{n}_{kC}} + \int_{C_k} \mathbf{n}_k \centerdot \mathbb{V}_k \centerdot \mathbf{n}_z \frac{\mathrm{d}l}{\mathbf{n}_k \centerdot \mathbf{n}_{kC}}$$

- Transfers are dominant in the radial direction, quasi fully developed flows,

$$\frac{\partial}{\partial z} A_k < \mathbf{n}_z \centerdot \mathbb{V}_k \centerdot \mathbf{n}_z >_2 \propto \frac{\partial}{\partial z} \left(\nu_t \frac{\partial w}{\partial z} \right) \to 0$$

- Give an example of a situation where this term might not be neglected.
- **Closures are required**: interactions at the interface and the wall (wall friction).

TIME AVERAGED BALANCE EQUATIONS



- Conditional time-averaging,

$$\overline{f}_k^X = \frac{1}{T_k}\int_{[T_k]} f_k \mathrm{d}t = \frac{\int_T X_k f_k \mathrm{d}t}{\int_T X_k \mathrm{d}t} = \frac{\overline{X_k f_k}}{\overline{X_k}}$$

- Plain time-average,

$$\overline{f} = \frac{1}{T}\int_T f \mathrm{d}t$$

- How to get a balance equation for a mean value? Average the local balance on $[T_k]$.

$$\int_{[T_k]} \frac{\partial \rho_k}{\partial t} \mathrm{d}t + \int_{[T_k]} \nabla \cdot (\rho_k \mathbf{v}_k)\, \mathrm{d}t = 0$$

- Limiting forms of the Leibniz rule and the Gauss, theorem,

$$\frac{\partial}{\partial t} \underbrace{\int_{[T_k]} \rho_k \mathrm{d}t}_{T_k \overline{\rho_k}^X} + \cdots + \nabla \cdot \underbrace{\int_{[T_k]} \rho_k \mathbf{v}_k \mathrm{d}t}_{T_k \overline{\rho_k \mathbf{v}_k}^X} + \cdots = 0$$

- Limiting form of the Leibniz rule (derivation of an integral wrt upper limit),

$$\int_{[T_k]} \frac{\partial f_k}{\partial t} \mathrm{d}t = \frac{\partial}{\partial t} \int_{[T_k]} f_k \mathrm{d}t - \sum_{\mathrm{disc.} \in [T]} f_k \underbrace{\frac{\mathbf{v}_i \cdot \mathbf{n}_k}{|\mathbf{v}_i \cdot \mathbf{n}_k|}}_{\pm 1}$$

- Limiting form of the Gauss theorem,

$$\int_{[T_k]} \nabla \cdot \mathbf{B}_k \mathrm{d}t = \nabla \cdot \int_{[T_k]} \mathbf{B}_k \mathrm{d}t + \sum_{\mathrm{disc.} \in [T]} \frac{\mathbf{n}_k \cdot \mathbf{B}_k}{|\mathbf{v}_i \cdot \mathbf{n}_k|}$$

- Time-averaged mass balance, production rate of phase k, interfacial interactions are homogenized [kg/s/m^3],

$$\frac{\partial \alpha_k \overline{\rho_k}^X}{\partial t} + \nabla \cdot \left(\alpha_k \overline{\rho_k \mathbf{v}_k}^X\right) = -\frac{1}{T} \sum_{\mathrm{disc.} \in [T]} \frac{\dot{m}_k}{|\mathbf{v}_i \cdot \mathbf{n}_k|}$$

- Time-averaged balance equations, starting point of the Reynolds decomposition, T?

$$\frac{\partial \alpha_k \overline{\rho_k \psi_k}^X}{\partial t} + \nabla \cdot \left(\alpha_k \overline{\rho_k \mathbf{v}_k \psi_k}^X\right) + \nabla \cdot \left(\alpha_k \overline{\mathbf{j}_{\psi k}}^X\right) - \alpha_k \overline{\phi_k}^X = -\frac{1}{T} \sum_{\mathrm{disc.} \in [T]} \frac{\dot{m}_k \psi_k + \mathbf{n}_k \cdot \mathbf{j}_{\psi k}}{|\mathbf{v}_i \cdot \mathbf{n}_k|}$$

COMPOSITE AVERAGES: THE TWO-FLUID MODEL

- Example: mass balance, space-averaged and time averaged,

$$\frac{\partial}{\partial t}\overline{A_k < \rho_k >_2} + \frac{\partial}{\partial z}\overline{A_k < \rho_k w_k >_2} = \overline{-\int_{C_k} \dot{m}_k \frac{\mathrm{d}l}{\mathbf{n}_k \,\centerdot\, \mathbf{n}_{kC}}}$$

- Time averaged and space averaged,

$$\frac{\partial}{\partial t} A \nless \alpha_k \overline{\rho_k}^X \ngtr_2 + \frac{\partial}{\partial z} A \nless \alpha_k \overline{\rho_k w_k}^X \ngtr_2 = -A \nless \frac{1}{T} \sum_{\mathrm{disc.} \in [T]} \frac{\dot{m}_k}{|\mathbf{v}_i \,\centerdot\, \mathbf{n}_k|} \ngtr_2$$

- LHS are identical, the RHS should also. Proof, identity on interaction terms,
- Local specific interfacial area, $\gamma = \frac{1}{T} \sum_{\mathrm{disc.} \in [T]} \frac{1}{|\mathbf{v}_i \,\centerdot\, \mathbf{n}_k|}$
- Possible closure of interaction terms: interfacial area $\times$ mean flux,

$$\frac{1}{T} \sum_{\mathrm{disc.} \in [T]} \frac{\dot{m}_k}{|\mathbf{v}_i \,\centerdot\, \mathbf{n}_k|} = \frac{\dot{m}_{ki}}{T} \sum_{\mathrm{disc.} \in [T]} \frac{1}{|\mathbf{v}_i \,\centerdot\, \mathbf{n}_k|} = \gamma \dot{m}_{ki}$$

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